Solution for Second-Order Differential Equation Using Least Square Method

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Abstract: This paper studies the numerical method for solving differential equations. The continuous least square method (CLSM) is used to obtain the explicit solution for solving ordinary differential equations (ODEs), partial differential equations (PDEs), and fractional differential equations (FDEs), but in this work, we consider the explicit results from CLSM approach and applied it on second-order ODEs. Moreover, the L_2 norm is used to obtain the minimum approximation error. The numerical results obtained has a good agreement with the exact solution with minimum approximation error. The explicit results are supported by an example that was treated with Matlab and Matematica 11.

Keywords: Differential Equation, Second-order Differential Equation, Continuous Least Square Method, and L_2 norm

1. Introduction

The CLSM is an important issue in solving ODEs, which play a great role in mathematical physics. The efforts of finding several methods for solving problems of ODEs have been practiced by many researchers (Eason, 1976; Loghmani, 2008; Katayoun, 2004). The discrete least square method (DLSM) and the CLSM are used to solve 2nd-order ODEs with minimum approximation errors (Ibrahim, 2020; Ibrahim & Ibrahim, 2022b). The CLSM approach is used to solve complex problems involving ODEs, and FDEs (Ibrahim & Isah, 2021; Isah & Ibrahim, 2021). The authors in (Ibrahim & Rababah, 2020; Rababah & Ibrahim, 2016a; Rababah & Ibrahim, 2016b; Rababah & Ibrahim, 2018) introduced a numerical approximation approach that involves curves and surfaces which play a vital role in numerical analysis. When it comes to the application perspective, the authors in (Ibrahim, 2022ba; Ibrahim & Koksal, 2021a; Ibrahim & Koksal, 2021b; Salisu, 2021; Salisu & Abedallah, 2022) make use of commutativity to study the relation and the sensitivity between systems, the idea can be extended to investigate the commutativity and sensitivity using the CLSM approach.

This paper aims to promote numerical techniques for (ODEs). The L_2 norm alongside the (CLSM) is used to obtain the minimum approximation error and numerical approximate solution, respectively.

2. Preliminaries

In this research, the CLSM for solving ODEs is considered.

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$$\mathrm{L(y) = f(x) for\ x \in domain\ \Omega,}$$
$$\mathrm{W(y) = g(x)\ for\ x \in domain\ } \delta\Omega.$$

Where L stands for the differential operator and Ω indicates the domain in R^1 or R^2 or R^3, while W refers to the boundary operator. The approximate solution of ODEs can be written as

$$\tilde{y} = \sum_{i=1}^{n} q_i\, C^i(X). \qquad [1]$$

$C^i(X)$ and q_i represent the weighted basis function and the coefficients (weights) respectively, the q_i is realized using the CLSM. Let the residual R_L(X) and R_W(X) be defined as

$$R_L(\mathrm{x}, \tilde{y}) \;=\; \mathrm{L}\,(\tilde{y}) - \mathrm{f(x)} \quad \mathrm{for} \quad \mathrm{x} \in domain\ \Omega\,. \qquad [2]$$

$$R_W(\mathrm{x}, \tilde{y}) = w\,(\tilde{y}) - g(x) \quad for\ \ x \in boundary\ \delta\Omega. \qquad [3]$$

Substituting y_{exact} into Eq. (2) and Eq. (3) leads to $R_L(\mathrm{x}, y_{exact}) = 0$ and $R_W(\mathrm{x}, y_{exact}) = 0$ respectively.

3. Material and Method

In this section we consider the CLSM, the CLSM is an approximation process that involves the use of L_2 norm to solve ODEs, the q_i from Eq. (1) is obtained using the CLSM, considering the Minimize error function as

$$E = \int_{\Omega} R_L^2\,(x, \tilde{y})dx + \int_{\alpha\Omega} R_W^2(x, \tilde{y})dx. \qquad [4]$$

The first derivative of Eq. (4) concerning q_i and equating to zero leads to

$$\frac{\partial E}{\partial q_i} = 0, \qquad for\ i = 1, \dots N,$$

which yields

$$\int_{\Omega} R_L(x, \tilde{y})\,\frac{\partial R_L}{\partial q_i}dx + \int_{\alpha\Omega} R_W(x, \tilde{y})\,\frac{\partial R_L}{\partial q_i}dx = 0 \quad i = 1, \dots, N. \qquad [5]$$

Eq. (5) is the algebraic equation that can be written in the form of

$$Ma = b \qquad [6]$$

Note that M is n x n matrix, $a = [q_1, q_2, q_3, \dots, q_n]$T, and some column vector b.

4. Results and Discussions

In this section, we make use of the results obtained from the CLSM and implement them by considering an example.

Example 1: Consider the 2nd-order initial value problem.

$$\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + \sqrt{5}\, y = 0, \qquad y(0) = 0.3, \qquad y^{I}(0) = 0.4, \tag{7}$$

where $0 \leq x \leq 1$. Let

$$L(x, y) = \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + \sqrt{5}\, y \tag{8}$$

Step 1: Let the polynomial.

$$\tilde{y} = \sum_{i=1}^{N} q_i\, x^i + y_0. \tag{9}$$

Step 2: We set $y_0 = 0.3$ and $q_1 = 0.4$ in Eq. (9) to satisfy the boundary condition.

Step 3: The residual

$$R(x) = \frac{d^2\tilde{y}}{dx^2} + (x+1)\frac{d\tilde{y}}{dx} + \sqrt{5}\, \tilde{y}. \tag{10}$$

By replacing $\tilde{y}(x)$ from Eq. (9) into Eq. (10), we will get:

$$R(x) = \frac{d^2\left(\sum_{i=1}^{N} q_i\, x_i + 0.3\right)}{dx^2} + (x+1)\frac{d\left(\sum_{i=1}^{N} q_i\, x_i + 0.3\right)}{dx} + \sqrt{5}\left(\sum_{i=1}^{N} q_i x_i + 0.3\right) \tag{11}$$

Step 4: The minimum error is obtained by considering

$$E = \int_0^1 R^2(x)dx. \tag{12}$$

Step 5: The continuous least-square solution is obtained by solving Eq. (12).

$$\frac{\partial E}{\partial q_i} = 0, \qquad for \quad i = 1, .., N, \tag{13}$$

$$\int_0^1 R(x)\frac{\partial R}{\partial q_i} dx = 0, \qquad i = 1, \dots, N. \tag{14}$$

Substituting Eq. (11) into Eq. (14) for N = 3, we obtain the following matrices with the help of the Matlab program

$$D = \begin{pmatrix} 45.6127 & 61.6099 \\ 61.6099 & 89.0385 \end{pmatrix}, \quad b = \begin{pmatrix} 16.5054 \\ 21.2004 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \end{pmatrix}. \tag{15}$$

Solving Eq. (15) lead to

$$q_1 = 0.4, q_2 = -0.615629, q_3 = 0.187878.$$

And the approximate solution is given as

$$\tilde{y} = 0.187878\, x^3 - 0.615629\, x^2 + 0.4\ \ x + 0.3. \qquad [16]$$

The exact solution is given by

$$y_{exact} = 0.2560850487909125 e^{-x-\frac{x^2}{2}} \left(1.\,\text{HermiteH}\left[-1+\sqrt{5}, \frac{1}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right] - 0.5189851135455146\text{Hypergeometric1F1}\left[\frac{1}{2}\left(1-\sqrt{5}\right), \frac{1}{2}, \left(\frac{1}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right)^2\right]\right). \qquad [17]$$

The approximate with exact solutions and error is depicted in Figure 1 and Figure 2 for $N = 3$

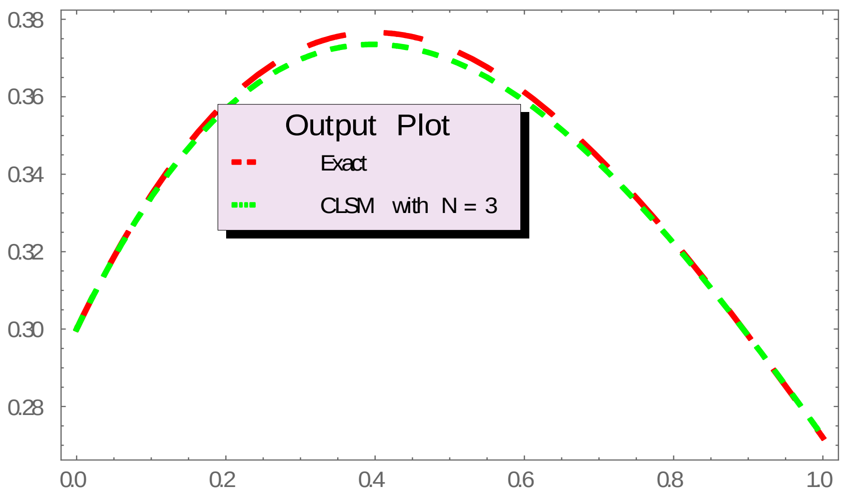

Figure 1: The result of example 1 with N = 3

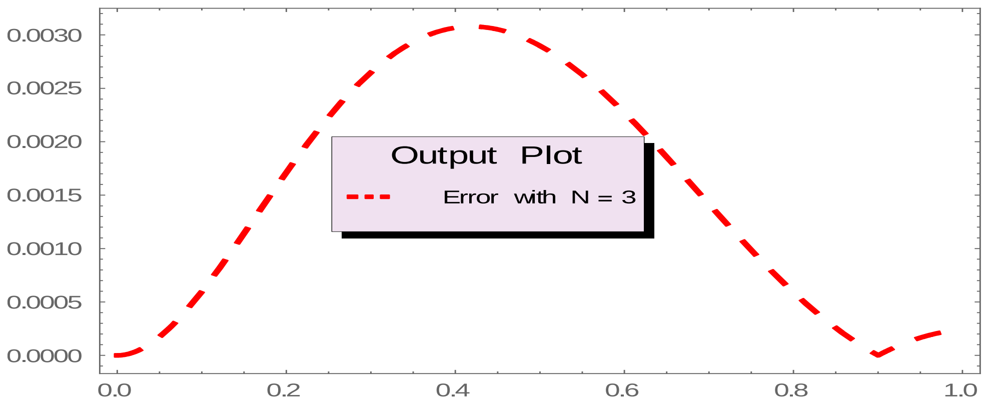

Figure 2: The error plots of Example 1 with N=3

For $N = 5$, we obtain the following matrices

$$D = \begin{pmatrix} 45.61 & 61.61 & 77.88 & 94.25 \\ 61.61 & 89.03 & 116.91 & 145.00 \\ 77.88 & 116.91 & 158.05 & 200.27 \\ 94.25 & 145.00 & 200.27 & 258.15 \end{pmatrix}, \quad b = \begin{pmatrix} 16.51 \\ 21.20 \\ 25.91 \\ 30.62 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}. \qquad [18]$$

And the approximate solution is

$$\tilde{y} = -0.0649883x^5 + 0.209667x^4 - 0.0367039x^3 - 0.535967x^2 + 0.4x + 0.3. \quad [19]$$

The approximate with exact solutions and the error plots are depicted in Figure 3 and Figure 4 respectively, for N =5

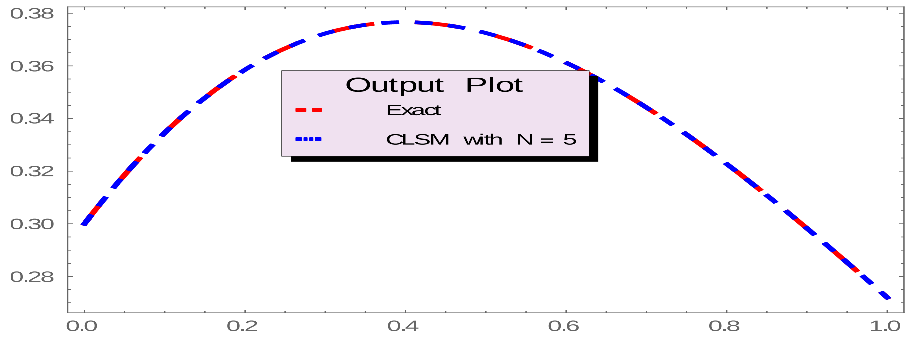

Figure 3: The results of example 1 with N = 5

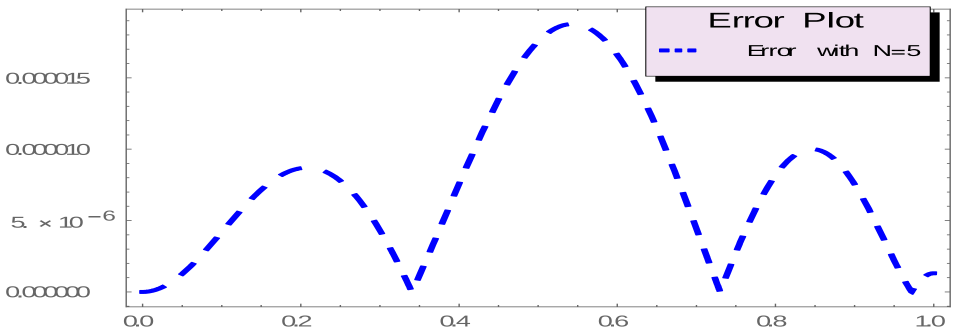

Figure 4: The Error plots of example 1 with N=6

The comparison between the exact CLSM and error are depicted in Figure 5 and Figure 6 for $N = 3$ and that of N = 5.

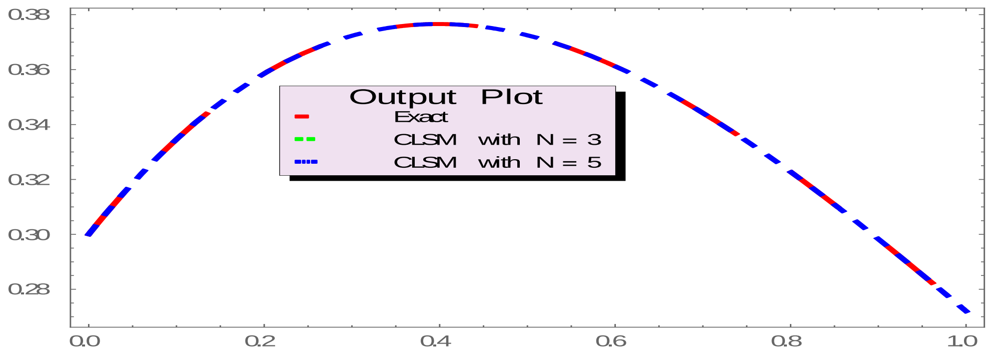

Figure 5: The results of example 1 with N = 3 and N=6

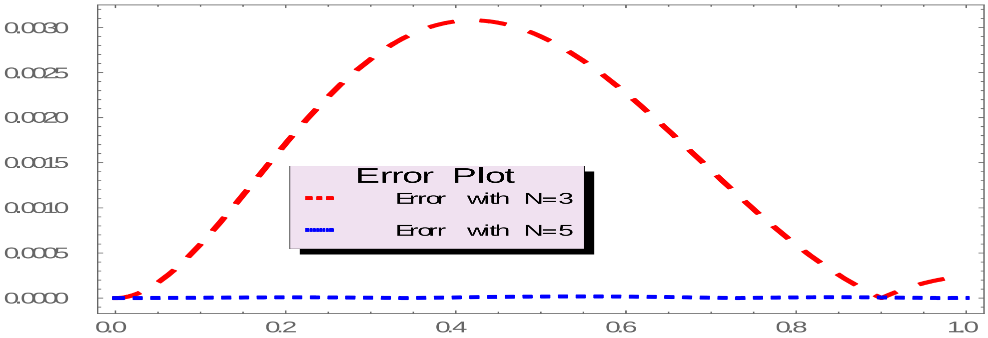

Figure 6: The error plots of example 1 with N=3 and N=6

Table 1: Data of errors analysis with N=3 and N=6 for 2nd - order ODE

x	Y exact	Y CLSM. with N = 3	y CLSM with N = 5	Errors with N = 3	Errors with N = 5
0	0.3	0.3	0.3	0	0
0.1	0.334628	0.334032	0.334624	0.000596543	4.18861×10^{-7}
0.2	0.358591	0.356878	0.358582	0.00171316	8.66736×10^{-6}
0.3	0.372317	0.36967	0.372312	0.00265057	4.31975×10^{-6}
0.4	0.376591	0.373524	0.376598	0.00306711	7.5654×10^{-6}
0.5	0.372476	0.369577	0.372494	0.00289852	0.0000175481
0.6	0.361227	0.358955	0.361243	0.00227139	0.0000166093
0.7	0.344201	0.342784	0.344205	0.00141683	4.45457×10^{-6}
0.8	0.322781	0.322191	0.322773	0.000590218	8.16031×10^{-6}
0.9	0.298305	0.298303	0.298297	1.2752×10^{-6}	7.54997×10^{-6}
1.0	0.272007	0.2722489	0.272008	0.000242287	1.30805×10^{-6}

5. Conclusion

This paper investigates a numerical method for solving differential equations. The (CLSM) together with the L_2 norm is used to find better approximation with the minimal error by solving differential equations, The results obtained are shown to be equivalent with the exact solution with minimum error. The results are supported with MATLAB and Wolfram Mathematica 11.

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