Eurasian Journal of Science & Engineering
ISSN 2414-5629 (Print), ISSN 2414-5602 (Online) EAJSE

Optimal Procedures for Certain Crossed Repeated Measures Model

Sami D. Gabbara[1] & Ghadah A. AL-Sakkal[2]

[1]Department of Mathematics, College of Science, Salahaddin University, Erbil, Iraq
[2]Department of Computer, Faculty of Engineering, Tishk International University, Erbil, Iraq
Correspondence: Ghadah A. AL-Sakkal, Tishk International University, Erbil, Iraq.
Email: ghada.alsakkal@ishik.edu.iq

doi: 10.23918/eajse.v5i1p1

Abstract: In this paper, we develop a general method to analyze several different kinds of certain crossed repeated measures models (CRMM) which represent many situations occurring in repeated measurements on the same experimental units (individuals). Let $Y_i = (Y_{111}, \ldots, Y_{idrc})'$ be the vector of observations of the i^{th} individuals. It is assumed that the Y_i are jointly normally distributed with mean μ_i . We want to test hypotheses about μ_i . In order to get powerful tests we make the simplifying assumptions that all measurements have the same variance σ^2 and every pair of measurements that comes from (i) different bulls and different cows (ii) different bulls but with the same cow (iii) the same bull with different cows; have covariance's $0, \sigma^2\rho_1, \sigma^2\rho_2$ respectively. And every pair of measurements that comes from the same bull and the same cow with treatments of (a) different columns and different rows (b) the same column but different rows (c) different columns but the same row have covariance's $\sigma^2\rho_3, \sigma^2\rho_4$ and $\sigma^2\rho_5$, respectively. The results of this model can be used to analyze certain 4-way balanced mixed and/or random effects models. This procedure is also useful to analyze any of the mentioned 4-way models by adding any number of fixed effects to the model as long as those added effects do not interact with any random effects already in these models.

Keywords: Coordinate-Free, Mixed Models, Random Models, Repeated Measures Models

1. Introduction

The crossed repeated measures models (CRMM) is one of the most widely used models in experimental design, especially in biological, agriculture, education and psychological research (see Lehman, 1959; Cox, 1992; Hoshmand, 2006). Arnold (1979) has developed a general method to analyze repeated measures model (RMM), when each of m independent individuals receives several treatments and assuming that all measurements have the same $\sigma^2\rho$ and every pair of measurements that comes from the same individual have covariance $\sigma^2\rho$ and each individual is normally distributed. Gabbara (1985) has extended the RMM of Arnold (1979) to (i) nested repeated measures models (NRMM), (ii) generalized nested repeated measures models (GNRMM), (iii) crossed repeated measures models (CRMM), (iv) crossed-nested repeated measures models (CNRMM). Rhonda, and et al (2016) considered covariance models to account for NRM and simultaneously address mean profile estimation with penalized splines via semi parametric regression with application to a prospective study of 24-hour ambulatory blood pressure and the impact of surgical intervention on obstructive sleep apnea.

In this paper, we have generalized the work of Arnold (1979) to a more complicated situation occurring in the analysis of variance (ANOVA) when a particular individual receives every pair of treatment levels, in which observations cannot be assumed independent as they are assumed in the usual independent RMM**.** Let $Y_{ijk\ell}$ be the observations of the $(k,\ell)^{th}$ treatment on the calf from the j^{th} cow and the i^{th} bull, where $1 \le i \le m,\ 1 \le j \le d,\ 1 \le k \le r,\ 1 \le \ell \le c$. Let $Y_i = (Y_{1111}, \ldots, Y_{idrc})'$ be the vector of observations of the i^{th} individuals. It is assumed that the Y_i are jointly normally distributed with mean μ_i. We want to test hypotheses about μ_i. One possible model for this problem would be that Σ is taken as an arbitrary positive definite matrix, but the procedures for such model would have low power. Therefore, in order to get powerful tests, we assume that all measurements have the same variance σ^2 and every pair of measurements that comes from **(i)** different bulls and different cows **(ii)** different bulls but with the same cow (**iii)** the same bull with different cows have covariance's $0,\ \sigma^2\rho_1,\ \sigma^2\rho_2$ respectively. And every pair of measurements that comes from the same bull and the same cow with treatments of **(a)** different columns and different rows (**b**) the same column but different rows **(c)** different columns but the same row have covariance's $\sigma^2\rho_3,\ \sigma^2\rho_4$ and $\sigma^2\rho_5$, respectively. In symbols

$$\operatorname{cov}(Y_{ijk\ell}, Y_{i'j'k'\ell'}) = \begin{cases} \sigma^2 & \text{if} \quad i=i',\ j=j',\ k=k',\ \ell=\ell' \\ \sigma^2\rho_5 & \text{if} \quad i=i',\ j=j',\ k=k',\ \ell\ne\ell' \\ \sigma^2\rho_4 & \text{if} \quad i=i',\ j=j',\ k\ne k',\ \ell=\ell' \\ \sigma^2\rho_3 & \text{if} \quad i=i',\ j=j',\ k\ne k',\ \ell\ne\ell' \\ \sigma^2\rho_2 & \text{if} \quad i=i',\ j\ne j' \\ \sigma^2\rho_1 & \text{if} \quad i\ne i',\ j=j' \\ 0 & \text{if} \quad i\ne i',\ j\ne j' \end{cases} \qquad \textbf{[1]}$$

Assuming that the design is given, we use a coordinate-free approach to find optimal (i.e, UMP invariant, UMP unbiased, most stringent, etc.) procedures for testing a large class of hypotheses about μ_i. For this model, we write

$$\mu = \begin{bmatrix} \mu_{1111} \\ \vdots \\ \mu_{mdrc} \end{bmatrix} = \beta_0 1_{mdrc} + 1_{crd} \otimes \beta_1 + 1_{cr} \otimes \beta_2 \otimes 1_m + 1_{cr} \otimes \beta_3 + 1_c \otimes \beta_4 + \beta_5^* + \beta_6 \qquad \textbf{[2]}$$

where $1_s = (1,\ldots,1)' \in R^s$, $s \ge 2$, $\otimes$ is the Kronecker product operation of two matrices, $\beta_0 \in R$ is the overall mean (grand mean), $\beta_1 = (\beta_1^1, \ldots \beta_1^m)' \in R^m$ is an $m \times 1$ vector orthogonal to 1_m (i.e. whose average is zero for each bull), $\beta_2 = (\beta_2^1, \ldots, \beta_2^d)' \in R^d$ is and $d \times 1$ vector orthogonal to 1_d (i.e. whose average is zero for each cow), $\beta_3 = (\beta_3^{11}, \ldots \beta_3^{md})' \in R^{md}$ is an $md \times 1$ vector orthogonal to every column of the matrix $I_d \otimes 1_m$ and every column of the matrix $I_d \times 1_m$ (i.e. whose average is zero for each bull and each cow), $\beta_4 = (\beta_4^{111}, \ldots, \beta_4^{mdr})' \in R^{mdr}$ is an $mdr \times 1$ vector orthogonal to every column of the matrix $1_r \otimes I_{md}$ (i.e. whose average is zero for each row treatment in a certain mating), and $\beta_5 = (\beta_5^{(11)'}, \ldots, \beta_5^{(md)'})' \in R^{mdc}$ where β_5 is an $mdc \times 1$ vector orthogonal to every column of the matrix $1_c \otimes I_{md}$ (i.e. whose average is zero for each column treatment in a certain mating) where $\beta_5^{(ij)} = (\beta_5^{ij1}, \ldots, \beta_5^{ijc})'$, $\beta_6 = (\beta_6^{1111}, \ldots \beta_6^{mdrc})' \in R^{mdrc}$ is an $mdrc \times 1$ vector orthogonal to every column of the matrix $1_c \otimes I_{mdr}$ and every column of the matrix $I_c \otimes 1_r \otimes I_{md}$ (i.e. whose average is zero for each column treatment and for each row treatment).

We consider testing hypotheses about β_h (type h), for $h = 1,\ldots,6$. We show that optimal test is an F-test. The sum of squares **(SS)** and the degrees of freedom **(df)** for effect being tested is the same as they would be if the measures were independent. However, the **SS** and **df** for denominator are different for the six types of problems. We will also discuss various hypotheses about the correlation coefficients which are 14. The problem studied in this paper transform to a product of more than two problems. Following Arnold 1973, we define recursively such a product by

$$P_1 \times \ldots \times P_s = (P_1 \times \ldots \times P_{s-1}) \times P_s$$

and the result valid for two products holds good for s products also. Saarinen F. (2004) gave an example for the mixed model and their use in repeated measurement. Baayen et al. (2008) worked simultaneous example for mixed effects modeling.

2. Setting Up the Model

2.1 Defining The Model

Let Y be an $mdrc$-dimensional random vector, such that $Y \sim N_{mdrc}(\mu,\Sigma)$, where μ is defined in (2) and using (1), Σ can be written as follows:

$$\Sigma = \sigma^2[(1-\rho_5-\rho_4+\rho_3)I_{mdrc} + (\rho_5-\rho_3)J_c \otimes I_{mdr} + (\rho_4-\rho_3)I_c \otimes J_r \otimes I_{md}$$
$$(\rho_3-\rho_2-\rho_1)J_{cr} \otimes I_{md} + \rho_2 J_{drc} \otimes I_m + \rho_1 J_{cr} \otimes I_d \otimes J_m] \qquad \textbf{[3]}$$

where $J_s = 1_s 1'$ be the $s \times s$ matrix of one's**.** We assume that $\Sigma > 0$ which is equivalent to

$$\sigma^2 > 0, \quad -\frac{1}{c-1} < \rho_5 < 1, \quad -\frac{1}{r-1} < \rho_4 < 1,$$
$$-\frac{1}{r(c-1)}[c+(r-c)\rho_4] < \rho_3 < \frac{1}{r(c-1)}[c+(cr-c-r)\rho_4],$$
$$-\frac{1}{r(d-1)}[1+(r-1)\rho_4] < \rho_2 < \frac{1}{r}[1+(r-1)\rho_4],$$
$$-\frac{1}{r(m-1)}[1+(r-1)\rho_4 < \rho_1 < \frac{1}{r}[1+(r-1)\rho_4].$$

(see Lemma 1)

Let U_s be the 1-dimensional subspace of R^s generated by 1_s**.** Then

$$\mu = \sum_{h=0}^{6} P_{L_h}\mu$$

where

$$L_0 = U_{mdrc},\ L_1 = U_{drc}^m | U_{mdrc},\ L_2 = U_{mrc}^d | U_{mdrc},\ L_3 = (U_{cr}^d | U_{drc})^m | (U_{mrc}^d | U_{mdrc})$$
$$L_4 = (U_c^r | U_{rc})^{md}, L_5 = (U_r^c | U_{rc})^{md},\ L_6 = [(U_r^{\perp})^c | (U_c^r | U_{rc})]^{md}$$

and P_{L_h} is the projection matrix of the subspace L_h, where h=0, …,6. So that

$$\beta_0 1_{mdrc} = P_{L_0}\mu, \quad 1_{crd} \otimes \beta_1 = P_{L_1}\mu, \quad 1_{cr} \otimes \beta_2 \otimes 1_m = P_{L_2}\mu$$
$$1_{cr} \otimes \beta_3 = P_{L_3}\mu, \quad 1_c \otimes \beta_4 = P_{L_4}\mu, \quad \beta_5^{*} = P_{L_5}\mu, \quad \beta_6 = P_{L_6}\mu$$

Hence, this representation for μ always exists and is unique. Therefore, the transformation from μ to β_h, $h=1,\ldots,6$ is just a re-parameterization of the problem. Let

$$Y_{ij}=\begin{bmatrix} Y_{ij11} \\ \vdots \\ Y_{ijrc} \end{bmatrix}, \quad Y=\begin{bmatrix} Y_{11} \\ \vdots \\ Y_{md} \end{bmatrix}, \quad Cov(Y)=\Sigma$$

In order to define the parameter space, let T_1 be a t_1-dimensional subspaces of R^m, such that $T_1 \subset U_m^{\perp}$; $t_1 < m-1$, let T_2 be a t_2-dimensional subspaces of R^d, such that $T_2 \subset U_d^{\perp}$; $t_2 < d-1$, let T_3 be a t_3-dimensional subspaces of R^{md}, such that $T_3 \subset (U_m^{\perp})^d \mid (U_d^m \mid U_{md})$; $t_3 < (m-1)(d-1)$, let T_4 be a t_4-dimensional subspaces of R^{mdr}, such that $T_4 \subset (U_r^{\perp})^{md}$, $t_4 < md(r-1)$, let T_5 be a t_5-dimensional subspaces of R^{mdc}, such that $T_5 \subset (U_c^{\perp})^{md}$; $t_5 < md(c-1)$, let T_6 be a t_6-dimensional subspaces of R^{mdrc}, such that $T_6 \subset [(U_r^{\perp})^c \mid (U_c^r \mid U_{rc})]^{md}$, $t_6 < md(r-1)(c-1)$. For this paper, it is assumed that the parameter space is given by

$$\beta_0 \in R,\ \beta_1 \in T_1,\ \beta_2 \in T_2,\ \beta_3 \in T_3,\ \beta_4 \in T_4,\ \beta_5 \in T_5,\ \beta_6 \in T_6,\ \Sigma > 0 \qquad \textbf{[4]}$$

The model defined by (1)-(4) is called the CRMM.
We consider twenty different hypotheses testing problems for this model. For all twenty problems the alternative set is the parameter space given in (4).

a. Let $Q_h \subset T_h$ be an q_h-dimensional subspace, $q_h < t_h$ for $h=1,\ldots,6$. In the h^{th} problem for $h=1,\ldots,6$ we test that

$$\beta_0 \in R,\ \beta_h \in Q_h,\ \beta_s \in T_s,\ s=1,\ldots,6,\ s \neq \ell .$$

b. The remaining fourteen problems are to test that

(1) $\rho_3=\rho_4$, **(2)** $\rho_3=\rho_4=0$, **(3)** $\rho_3=\rho_5$, **(4)** $\rho_3=\rho_5=0$, **(5)** $\rho_1=\rho_2$ **(6)** $\rho_3=0,\ \rho_4=\rho_5$, **(7)** $\rho_1=\rho_2=0$, **(8)** $\rho_1=\rho_2=\rho_3=0$, **(9)** $\rho_1=\rho_2=\rho_4=0$, **(10)** $\rho_1=\rho_2=\rho_5=0$, **(11)** $\rho_1=\rho_2=\rho_4=\rho_5=0$, **(12)** $\rho_3=\rho_4=\rho_5$ **(13)** $\rho_3=\rho_4=\rho_5=0$, **(14)** $\rho_1=\rho_2=\rho_3=\rho_4=\rho_5=0$.

2.2 Transforming The Model

In this section, we show how to transform the model defined in Section (2.1) to a model that is easier to handle. Let C_s' be an $(s-1)\times s$ orthonormal basis matrix for the sub-space $U_s^{\perp}$ such that

$$C_s'C_s = I_{s-1},\ C_s'1_s=0,\ 1_s'C_s=0,\ C_sC_s'=N_s=I_s-M_s,\ M_s=(1/s)J_s$$

Then

$$\Gamma = \begin{bmatrix} (mdrc)^{-1/2} & 1_c' \otimes 1_r' \otimes 1_d' \otimes 1_m' \\ (drc)^{-1/2} & 1_c' \otimes 1_r' \otimes 1_d' \otimes C_m' \\ (mrc)^{-1/2} & 1_c' \otimes 1_r' \otimes C_d' \otimes 1_m' \\ (rc)^{-1/2} & 1_c' \otimes 1_r' \otimes C_d' \otimes C_m' \\ (c)^{-1/2} & 1_c' \otimes C_r' \otimes \Gamma_d \otimes \Gamma_m \\ (r)^{-1/2} & C_c' \otimes 1_r' \otimes \Gamma_d \otimes \Gamma_m \\ & C_c' \otimes C_r' \otimes \Gamma_d \otimes \Gamma_m \end{bmatrix} = \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix}$$

is an $mdrc \times mdrc$ orthogonal matrix. Let

$$Y^* = \Gamma Y = \begin{bmatrix} D_0 Y \\ D_1 Y \\ D_2 Y \\ D_3 Y \\ D_4 Y \\ D_5 Y \\ D_6 Y \end{bmatrix} = \begin{bmatrix} Y_0^* \\ Y_1^* \\ Y_2^* \\ Y_3^* \\ Y_4^* \\ Y_5^* \\ Y_6^* \end{bmatrix}$$

where Y_0^* is an 1×1 vector, Y_1^* is an $(m-1)\times 1$ vector, Y_2 is an $(d-1)\times 1$ vector, Y_3 is an $(m-1)(d-1)\times 1$ vector, Y_4 is an $md(r-1)\times 1$ vector, Y_5 is an $md(c-1)\times 1$ vector and Y_6 is an $md(r-1)(c-1)\times 1$ vector.

Since, Γ is an invertible matrix and does not depend on any unknown parameter, then observing Y is equivalent to observing $Y_0^*, Y_1^*, Y_2^*, Y_3^*, Y_4^*, Y_5^*, Y_6^*$. Let

$$\tau_h^2 = \sigma^2 \lambda_h , \qquad h = 0, \ldots .6$$

where

$$\begin{aligned}
\lambda_0 &= [1+(c-1)\rho_5 + (r-1)\rho_4 + (c-1)(r-1)\rho_3 + cr(d-1)\rho_2 + cr(m-1)\rho_1 \\
\lambda_1 &= [1+(c-1)\rho_5 + (r-1)\rho_4 + (c-1)(r-1)\rho_3 + cr(d-1)\rho_2 - cr\rho_1] \\
\lambda_2 &= [1+(c-1)\rho_5 + (r-1)\rho_4 + (c-1)(r-1)\rho_3 - cr\rho_2 + cr(m-1)\rho_1] \\
\lambda_3 &= [1+(c-1)\rho_5 - (r-1)\rho_4 + (c-1)(r-1)\rho_3 - cr\rho_2 - cr\rho_1] \\
\lambda_4 &= [1+(c-1)\rho_5 - \rho_4 - (c-1)\rho_3] \\
\lambda_5 &= [1-\rho_5 + (r-1)\rho_4 - (r-1)\rho_3] \\
\lambda_6 &= [1-\rho_5 - \rho_4 - \rho_3]
\end{aligned} \qquad \textbf{[5]}$$

$(\tau_1^2, \tau_2^2, \tau_3^2, \tau_4^2, \tau_5^2, \tau_6^2)$, is just an invertible function of $(\sigma^2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$, which is a re-parameterization. We now find the joint distribution of Y^*.

Lemma 1. The random vectors $Y_0^*, Y_1^*, Y_2^*, Y_3^*, Y_4^*, Y_5^*, Y_6^*$ are independent and

$$Y_0^* \sim N_1((mdrc)^{1/2}\beta_0, \tau_0^2), \qquad Y_1^* \sim N_{(m-1)}((drc)^{1/2} C_m' \beta_1, \tau_1^2 I)$$

$$Y_2^* \sim N_{(d-1)}((mrc)^{1/2} C_d' \beta_2, \tau_2^2 I), \qquad Y_3^* \sim N_{(m-1)(c-1)}((rc)^{1/2}(C_d' \otimes C_m')\beta_3, \tau_3^2 I)$$

$$Y_4^* \sim N_{md(r-1)}(c^{1/2}(C_r' \otimes \Gamma_d \otimes \Gamma_m)\beta_4, \tau_4^2 I), \quad Y_5^* \sim N_{md(c-1)}(r^{1/2}(C_c' \otimes \Gamma_d \otimes \Gamma_m)\beta_5, \tau_5^2 I$$

$$Y_6^* \sim N_{md(r-1)(c-1)}((C_c' \otimes C_r' \otimes \Gamma_d \otimes \Gamma_m)\beta_6, \tau_6^2 I)$$

and $\sum > 0$ if and only if $\sigma^2 > 0$ and $\tau_h^2 > 0$ for all $h = 0,1,\ldots,6$.

Proof. It can be shown easily that

$$Y^* = \Gamma\, Y \sim N_{mdrc}(\mu^*, \Sigma^*)$$

where

$$\mu^* = \Gamma\mu = \begin{bmatrix} (mdrc)^{1/2}\,\beta_0 \\ (drc)^{1/2}\, C_m'\,\beta_1 \\ (mrc)^{1/2}\, C_d'\,\beta_2 \\ (rc)^{1/2}(C_d' \otimes C_m')\beta_3 \\ c^{1/2}(C_r' \otimes \Gamma_d \otimes \Gamma_m)\beta_4 \\ r^{1/2}(C_c' \otimes \Gamma_d \otimes \Gamma_m)\beta_5 \\ (C_c' \otimes C_r' \otimes \Gamma_d \otimes \Gamma_m)\beta_6 \end{bmatrix} \qquad \textbf{[6]}$$

and

$$\sum^* = \Gamma \sum \Gamma' = \text{diag}\,(\tau_0^2,\ \tau_1^2 I_{m-1},\ \tau_2^2 I_{d-1},\ \tau_3^2 I_{(m-1)(d-1)}, \tau_4^2 I_{md(r-1)},\ \tau_5^2 I_{md(c-1)}, \tau_6^2 I_{md(r-1)(c-1)})$$

Hence, $Y_0^*, Y_1^*, Y_2^*, Y_3^*, Y_4^*, Y_5^*, Y_6^*$ are independent. Therefore, the result follows.

Lemma 2. $\beta_0 = \bar{\mu}_{....}, \quad \beta_1^{(i)} = \bar{\mu}_{i...} - \bar{\mu}_{....}, \quad \beta_2^{(j)} = \bar{\mu}_{.j..} - \bar{\mu}_{....}, \quad \beta_3^{(ij)} = \bar{\mu}_{ij..} - \bar{\mu}_{.j..} - \bar{\mu}_{i...} + \bar{\mu}_{....}$

$$\beta_4^{(ijk)} = \bar{\mu}_{ijk.} - \bar{\mu}_{ij..}, \quad \beta_5^{(ij\ell)} = \bar{\mu}_{ij.\ell} - \bar{\mu}_{ij..}, \quad \beta_6^{(ijk\ell)} = \bar{\mu}_{ijk\ell} - \bar{\mu}_{ij.\ell} - \bar{\mu}_{ijk.} + \bar{\mu}_{ij..}$$

Proof. The result follows directly from (2) and (6).

Finally, we reparametrize the model. Let τ_h^2 be as defined in (5) and define

$$\gamma_0 = \sqrt{mdrc}\beta_0 \qquad \gamma_4 = \sqrt{c}(C_r' \otimes \Gamma_d \otimes \Gamma_m)\beta_4$$
$$\gamma_1 = \sqrt{drc}C_m'\beta_1 \qquad \gamma_5 = \sqrt{r}(C_c' \otimes \Gamma_d \otimes \Gamma_m)\beta_5$$
$$\gamma_2 = \sqrt{mrc}C_d'\beta_2 \qquad \gamma_6 = (C_c' \otimes C_r' \otimes \Gamma_d \otimes \Gamma_m)\beta_6\}$$
$$\gamma_3 = \sqrt{rc}(C_d' \otimes C_m')\beta_3$$

[7]

Then $\gamma_0 \in R$ if and only if $\beta_0 \in R$, $\gamma_h \in T_h^*$ if and only if $\beta_h \in T_h$, $h = 1,\ldots,6$

$\mu^* \in V^*$ if and only if $\mu \in V$, $\sum > 0$ if and only if $\tau_h^2 > 0 \quad h = 0,\ldots,6$

Corollary 3. The transformation from $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \sigma^2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5)$ to $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \tau_1^2, \tau_2^2, \tau_3^2, \tau_4^2, \tau_5^2, \tau_6^2)$ is an invertible function. Hence $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \tau_1^2, \tau_2^2, \tau_3^2, \tau_4^2, \tau_5^2, \tau_6^2)$ is just a re-parameterization of the problem. Therefore, we have transformed the CRMM to a model in which we observe $Y_0^*, Y_1^*, Y_2^*, Y_3^*, Y_4^*, Y_5^*, Y_6^*$ independent such that

$$Y_0^* \sim N_1(\gamma_0, \tau_0^2 I) \qquad Y_5^* \sim N_{md(c-1)}(\gamma_5, \tau_5^2 I)$$
$$Y_1^* \sim N_{(m-1)}(\gamma_1, \tau_1^2 I) \qquad Y_4^* \sim N_{md(r-1)}(\gamma_4, \tau_4^2 I)$$
$$Y_2^* \sim N_{(d-1)}(\gamma_2, \tau_2^2 I) \qquad Y_6^* \sim N_{md(r-1)(c-1)}(\gamma_6, \tau_6^2 I)$$
$$Y_3^* \sim N_{(m-1)(c-1)}(\gamma_3, \tau_3^2 I) \qquad \textbf{[8]}$$

We note that C_m , C_d , $C_d \otimes C_m$, $C_r \otimes \Gamma'_d \otimes \Gamma'_m$, $C_c \otimes \Gamma'_d \otimes \Gamma'_m$, $C_c \otimes C_r \otimes \Gamma'_d \otimes \Gamma'_m$ are orthonormal basis matrices for $U_m^{\perp}$, $U_d^{\perp}$, $(U_m^{\perp})^d \,|\, (U_d^m \,|\, U_{md})$, $(U_r^{\perp})^{md}$, $(U_c^{\perp})^{md}$, $[(U_r^{\perp})^c \,|\, (U_c^r \,|\, U_{rc})]^{md}$, respectively. Therefore $\dim V^* = \dim V$, $\dim T_h^* = \dim T_h$ for $h = 0,\ldots,6$. Now, if Q_h^* is a sub-space of T_h^* for $h = 1,\ldots,6$, Let

$$Q_1^* = \{C'_m u;\ u \in Q_1\} \qquad Q_4^* = \{(C'_r \otimes \Gamma_d \otimes \Gamma_m)u;\ u \in Q_4\}$$
$$Q_2^* = \{C'_d u;\ u \in Q_2\} \qquad Q_5^* = \{(C'_c \otimes \Gamma_d \otimes \Gamma_m)u;\ u \in Q_5\}$$
$$Q_3^* = \{(C'_d \otimes C'_m)u;\ u \in Q_3\} \qquad Q_6^* = \{(C'_c \otimes C'_r \otimes \Gamma_d \otimes \Gamma_m)u;\ u \in Q_6\}$$

The following lemma follows directly from the definitions.

Lemma 3:

a) If Q_h is a subspace of T_h, then $\beta_h \in Q_h$ iff $\gamma_h \in Q_h^*$ for all $h = 1,\ldots,6$

b) $$\sigma^2 > 0,\ -\frac{1}{c-1} < \rho_5 < 1,\ -\frac{1}{r-1} < \rho_4 < 1,$$
$$-\frac{1}{r(c-1)}[c + (r-c)\rho_4] < \rho_3 < \frac{1}{r(c-1)}[c + (cr - c - r)\rho_4],$$
$$-\frac{1}{r(d-1)}[1 + (r-1)\rho_4] < \rho_2 < \frac{1}{r}[1 + (r-1)\rho_4]$$
$$-\frac{1}{r(m-1)}[1 + (r-1)\rho_4] < \rho_1 < \frac{1}{r}[1 + (r-1)\rho_4]$$
iff $\tau_1^2 + \tau_2^2 - \tau_3^2 > 0,\ \tau_h^2 > 0$ for $h = 0,1,\ldots,6$ and

1. $\rho_3 = \rho_4$ *iff* $\tau_5^2 = \tau_6^2$
2. $\rho_3 = \rho_4 = 0$ *iff* $\tau_5^2 = \tau_6^2$
3. $\rho_3 = \rho_5$ *iff* $\tau_4^2 = \tau_6^2$
4. $\rho_3 = \rho_5 = 0$ *iff* $\tau_4^2 = \tau_6^2$
5. $\rho_1 = \rho_2$ *when* $d = m$ *iff* $\tau_1^2 = \tau_2^2$
6. $\rho_3 = 0,\ \rho_4 = \rho_5$ *when* $r = c$ *iff* $\tau_4^2 = \tau_5^2$
7. $\rho_1 = \rho_2 = 0$ *iff* $\tau_1^2 = \tau_2^2 = \tau_3^2$
8. $\rho_1 = \rho_2 = \rho_3 = 0$ *iff* $\tau_1^2 = \tau_2^2 = \tau_3^2$
9. $\rho_1 = \rho_2 = \rho_4 = 0$ *iff* $\tau_1^2 = \tau_2^2 = \tau_3^2$
10. $\rho_1 = \rho_2 = \rho_5 = 0$ *iff* $\tau_1^2 = \tau_2^2 = \tau_3^2$
11. $\rho_1 = \rho_2 = \rho_4 = \rho_5 = 0$ *iff* $\tau_1^2 = \tau_2^2 = \tau_3^2$
12. $\rho_3 = \rho_4 = \rho_5$ *iff* $\tau_4^2 = \tau_5^2 = \tau_6^2$
13. $\rho_3 = \rho_4 = \rho_5 = 0$ *iff* $\tau_4^2 = \tau_5^2 = \tau_6^2$
14. $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = 0$ *iff* $\tau_1^2 = \ldots = \tau_6^2$

3. Optimal Procedures

3.1 Calculating The Statistics

In this section, we find statistics F_h for $h = 1,\ldots,6$, that we need to do the required tests. Let

$$\begin{aligned}
&df_1 = (m-1) - t_1 = \dim(U_m^{\perp} \mid T_1), \quad df_2 = (d-1) - t_2 = \dim(U_d^{\perp} \mid T_2) \\
&df_3 = (m-1)(d-1) - t_3 = \dim([(U_m^{\perp})^d \mid (U_d^m \mid U_{md})] \mid T_3) \\
&df_4 = md(r-1) - t_4 = \dim[(U_r^{\perp})^{md} \mid T_4], \quad df_5 = md(c-1) - t_5 = \dim[(U_c^{\perp})^{md} \mid T_5] \\
&df_6 = md(r-1)(c-1) - t_6 = \dim\{[(U_r^{\perp})^c \mid (U_c^r \mid U_{rc})]^{md} \mid T_6\}
\end{aligned}$$

and define

$$\begin{aligned}
SS_1 &= df_1 M_1 = drc \sum_i (\bar{Y}_{i...} - \bar{Y}_{....} - \hat{\beta}_1^{(i)})^2 \\
SS_2 &= df_2 M_2 = mrc \sum_j (\bar{Y}_{.j..} - \bar{Y}_{....} - \hat{\beta}_2^{(j)})^2 \\
SS_3 &= df_3 M_3 = rc \sum_i \sum_j (\bar{Y}_{ij..} - \bar{Y}_{.j..} - \bar{Y}_{i...} + \bar{Y}_{....} - \hat{\beta}_3^{(ij)})^2 \\
SS_4 &= df_4 M_4 = c \sum_i \sum_j \sum_k (\bar{Y}_{ijk.} - \bar{Y}_{ij..} - \hat{\beta}_4^{(ijk)})^2 \\
SS_5 &= df_5 M_5 = r \sum_i \sum_j \sum_{\ell} (\bar{Y}_{ij.\ell} - \bar{Y}_{ij..} - \hat{\beta}_5^{(ij\ell)})^2 \\
SS_6 &= df_6 M_6 = \sum_i \sum_j \sum_k \sum_{\ell} (\bar{Y}_{ijk\ell} - \bar{Y}_{ij.\ell} - \bar{Y}_{ijk.} + \bar{Y}_{ij..} - \hat{\beta}_6^{(ijk\ell)})^2
\end{aligned}$$

3.2 Optimal Tests

In this section, we consider the testing problems mentioned in Section 2.1 and give the proof for the first and fifth problems as the proof for other can be done similarly.

3.2.1 Tests Concerning

β_1**.** We first look at the problem of type 1, in which we are testing $\beta_1 \in Q_1$ against $\beta_1 \in T_1$, so in the transformed model, we are testing that $\gamma_1 \in Q_1^*$ against $\gamma_1 \in T_1^*$ for the OLM involving only Y_1^* are independent having the distribution given in [8] and we are testing.

$$H_0 : \gamma_1 \in Q_1, \ \ \tau_1^2 > 0, \ \ \gamma_h \in T_h^*, \ h = 2,\ldots,6, \ \ , \ \ \tau_s^2 > 0, \ s =,2,\ldots,6$$

$$H_1 : \gamma_1 \in T_1, \ \ \tau_1^2 > 0, \ \ \gamma_h \in T_h^*, \ h = 2,\ldots,6, \ \ , \ \ \tau_s^2 > 0, \ s =,2,\ldots,6$$

Call this problem P . P is then the product of the testing problem $P_1,\ldots,P_6$. Where P_1 is the independent measures model in which we observe $Y_1^* \sim N_{(m-1)}(\gamma_1, \tau_1^2 I)$ and we are testing,

$$H_0 : \gamma_1 \in Q_1, \ \ \tau_1^2 > 0 \ \ \text{vs.} \ \ H_1 : \gamma_1 \in T_1, \ \ \tau_1^2 > 0$$

And P_i is the trivial problem [3], in which we observe $Y_i^* \sim N(\gamma_i, \tau_i^2 I)$ and we are testing

$$H_0 : \gamma_1 \in T_i^*, \;\; \tau_i^2 > 0 \;\; \text{vs.} \;\; H_1 : \gamma_1 \in T_i^*, \;\; \tau_i^2 > 0 \;\; for \;\; i =,2,\ldots,6$$

Since $P_2,\ldots,P_6$ are trivial problems (and hence the product of $P_2,\ldots,P_6$ is trivial [3], a good procedure for P_1 will be good for P . Therefore, let F_1 (see (10)) be the usual F_1 statistic and ϕ_1 the usual F critical function for testing P_1 such that

$$\phi_1(F_1) = \begin{cases} 1 & if \quad F_1 > F^{\alpha}_{t_1-q_1,\, df_1} \\ 0 & if \quad F_1 \le F^{\alpha}_{t_1-q_1,\, df_1} \end{cases}$$

where $F^{\alpha}_{t_1-q_1,\, df_1}$ is the upper α point of a central F distribution with $t_1 - q_1$ and df_1 degrees of freedom. We note that ϕ_1 would be the UMP invariant size α test for testing that $\gamma_1 \in Q_1^*$ against $\gamma_1 \in T_1^*$ for the OLM consisting only Y_1^*. It is also a UMP invariant size α test for the CRMM.

Theorem 5. $$F_1 \sim F_{t_1-q_1, df_1}\left[\frac{\left\| P_{T_1^*|Q_1^*}\gamma_1 \right\|^2}{\tau_1^2}\right]$$ **[9]**

The test ϕ_1 is size α , UMP invariant, UMP unbiased, most stringent, admissible, Bayes, and LRT test for P .

PROOF: the test has all these properties for P_1 [4], so it has these proportions for P by theorem B of [3].

3.2.2 Tests about

β_h , $h = 2,\ldots,6$. Follow in a similar way to section 3.2.1.

3.2.3 Testing that $\rho_3 = \rho_4$

Now, consider the problem of testing that $\rho_3 = \rho_4$ After transforming to $Y_0^*, Y_1^*,\ldots, Y_6^*$, this problem becomes the problem in which we observe $Y_0^*, Y_1^*,\ldots, Y_6^*$, independent and normally distributed as given in (8). We are testing

$$H_0 : \gamma_0 \in R, \;\; \gamma_h \in T_h^*, \;\; h = 1,\ldots,6, \;\; \tau_5^2 = \tau_6^2, \;\; \tau_s^2 > 0, \;\; s = 1,2,3,4$$

$$H_1 : \gamma_0 \in R, \quad \gamma_h \in T_h^*, \quad \tau_h^2 > 0, \quad h = 1,\ldots,6$$

This problem is not a product of problems. However, it is already a problem about what is known. It is the problem in which we have seven independent OLM's and are testing for the equality of two variances. Let

$$\phi_{a,b}(F_{5,6}) = \begin{cases} 1 & if \quad F_{5,6} > F^{b}_{df_5,df_6} \quad or \quad F_{5,6} < F^{1-a}_{df_5,df_6} \\ 0 & if \quad F^{1-a}_{df_5,df_6} \le F_{5,6} \le F^{b}_{df_5,df_6} \end{cases}$$

Where $a+b=\alpha$. Then, $\phi_{a,b}$ is a size α test. There is no UMP invariant test for this problem, but there exists a and b such that $\phi_{a,b}$ is a UMP unbiased, and other a and b such that $\phi_{a,b}$ is a LRT. The choice $a=b=\dfrac{\alpha}{2}$ is the choice used more often. In a similar way, we test $\rho_3=\rho_4=0$ (because this case in the CRMM), so, we test $\tau_5^2=\tau_6^2$ in the transformed model, i.e. we are testing for equality of two variances.

3.2.4 Testing that $\rho_3=\rho_5$

Follow in a similar way to section 3.2.3

3.2.5 Testing that $\rho_1=\rho_2=0$

Firstly, we consider testing the problem that $\rho_1=\rho_2=0$. In the transformed model that becomes the problem of testing that $\tau_1^2=\tau_2^2=\tau_3^2$. Therefore, this problem transforms to a problem of testing equality of variances in the three different OLM's involving Y_1^*, Y_2^* *and* Y_3^*. Hence, we are testing

$$H_0: \gamma_0 \in R,\ \ \gamma_h \in T_h^*,\ \ h=1,\dots,6,\ \ \tau_1^2=\tau_2^2=\tau_3^2,\ \ \tau_4^2>0,\ \tau_5^2>0,\ \tau_6^2>0$$

$$H_1: \gamma_0 \in R,\ \ \gamma_h \in T_h^*,\ \ \ \tau_h^2>0,\ \ \ h=1,\dots,6$$

There is no UMP invariant size α test for this problem, but an approximate size α test can be found by using Bartlett's test

4. Applications

In this section, we study six applications to illustrate the different types of hypotheses used (Al-Sakkal 1999).

Application 1. Assume a cattle breeding experiment in which we have m bulls and d cows with only one calf for each mating of bull and cow. We will consider a 2-way ANOVA model (with no interaction), with r row treatments and c column treatments such that each calf receives every pair of treatment levels. That is, each calf receives rc different treatment combinations. Let $Y=(Y_{1111},\dots,Y_{mdrc})$ be the vector of observations on all calves. Then $\mu_{ijk\ell}=\theta+\alpha_k+\gamma_\ell$, $\sum_k \alpha_k=0$, $\sum_\ell \gamma_\ell=0$, ($\mu_{ijk\ell}$ does not depend on i and j). We want to test that $\alpha_k=0$, and we want to test that $\gamma_\ell=0$, According to the lemma 2, $\beta_0=\theta,\ \beta_1^{(i)}=0,\ \beta_2^{(j)}=0,\ \beta_3^{(ij)}=0,\ \beta_4^{(ijk)}=\alpha_k,\ \beta_5^{(ij\ell)}=\gamma_\ell,$ $\beta_6^{(ijk\ell)}=0$

Therefore, the first hypothesis is of type 4, the second hypothesis is of type5.

Application 2. In this application. we will consider the 2-way model of application 1 but with interaction between the α effect and the γ effect, Then $\mu_{ijk\ell} = \theta + \alpha_k + \gamma_\ell + (\alpha\gamma)_{k\ell}$, $\sum_k \alpha_k = 0,\ \sum_\ell \gamma_\ell = 0,\ \sum_k (\alpha\gamma)_{k\ell} = \sum_\ell (\alpha\gamma)_{k\ell} = 0$ ($\mu_{ijk\ell}$ does not depend on i and j). We want to test that $\alpha\gamma_{k\ell} = 0$. According to the lemma 2, $\beta_6^{(ijk\ell)} = \alpha\gamma_{k\ell}$. We note that this hypothesis is like type 6.

Application 3. Consider a 3-way fixed effects in which model we have m bulls and d cows where each cow receives treatment, (h^{th}, $h = 1,\ldots,p$) during the pregnancy period and then we give each calf a combination of k^{th} row and ℓ^{th} column treatment levels $k = 1,\ldots,r;\ \ell = 1,\ldots,c$ .Then $\mu_{ijk\ell} = \theta + \alpha_h + \gamma_k + \eta_\ell + (\alpha\gamma)_{hk} + (\alpha\eta)_{h\ell} + (\gamma\eta)_{k\ell} + (\alpha\gamma\eta)_{hk\ell}$

$$\sum_h \alpha_h = 0,\ \sum_k \gamma_k = 0,\ \sum_\ell \eta_\ell = 0,\ \sum_h (\alpha\gamma)_{hk} = \sum_k (\alpha\gamma)_{hk} = 0,\ \sum_h (\alpha\eta)_{h\ell} = \sum_\ell (\alpha\eta)_{h\ell} = 0$$
$$\sum_k (\gamma\eta)_{k\ell} = \sum_\ell (\gamma\eta)_{k\ell} = 0,\ \sum_h (\alpha\gamma\eta)_{hk\ell} = \sum_k (\alpha\gamma\eta)_{hk\ell} = \sum_\ell (\alpha\gamma\eta)_{hk\ell} = 0$$

$\mu_{ijk\ell}$ does not depend on i and j . From Lemma (2-2), we see that

$$\beta_0 = \theta,\quad \beta_1^{(i)} = 0,\quad \beta_2^{(jh)} = \alpha_h,\quad \beta_3^{(ijh)} = 0,\quad \beta_4^{(ijhk)} = \gamma_k + (\alpha\gamma)_{hk}$$
$$\beta_5^{(ijh\ell)} = \eta_\ell + (\alpha\eta)_{h\ell},\quad \beta_6^{(ijk\ell)} = (\gamma\eta)_{\ell k} + (\alpha\gamma\eta)_{hk\ell}$$

We want to test that $\alpha_h = 0$. We note that this hypothesis is of type 2, we want to test that $\gamma_k = 0$. We note that this hypothesis is of type 4, we want to test that $\eta_\ell = 0$. We note that this hypothesis is of type 5, we want to test that $(\alpha\gamma)_{hk} = 0$. We note that this hypothesis is of type 4, we want to test that $(\alpha\eta)_{h\ell} = 0$. We note that this hypothesis is of type 5, we want to test that $(\gamma\eta)_{k\ell} = 0$. We note that this hypothesis is of type 6, we want to test that $(\alpha\gamma\eta)_{hk\ell} = 0$. We note that this hypothesis is of type 6.

Application 4. We now consider the balanced 4-way random effects model in which we have the first two effects interact and the third and fourth effect nested in the interaction of the first two we observe

$$Y_{ijk\ell} = \theta + a_i + b_j + (ab)_{ij} + c_{ijk} + d_{ij\ell} + e_{ijk\ell}$$

$$a_i \sim N(0,\sigma_a^2),\quad b_j \sim N(0,\sigma_b^2),\quad (ab)_{ij} \sim N(0,\sigma_{ab}^2)$$

$$c_{ijk} \sim N(0,\sigma_c^2),\quad d_{ij\ell} \sim N(0,\sigma_d^2),\quad e_{ijk\ell} \sim N(0,\sigma_e^2)$$

The parameter space for this model is given by

$$-\infty < \theta < \infty,\quad \sigma_a^2 \ge 0,\quad \sigma_b^2 \ge 0,\quad \sigma_{ab}^2 \ge 0,\quad \sigma_c^2 \ge 0,\quad \sigma_d^2 \ge 0,\quad \sigma_e^2 > 0$$

We are interested in testing that the $\sigma_a^2 = \sigma_b^2$, $\sigma_a^2 = \sigma_b^2 = 0$, $\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0$, $\sigma_c^2 = 0$, $\sigma_d^2 = 0$ and $\sigma_c^2 = \sigma_d^2$. We note that the $Y_{ijk\ell}$ and $Y_{i'j'k'\ell'}$ are not independent for the random effect model the $\operatorname{cov}(Y_{ijk\ell}, Y_{i'j'k'\ell'})$ is the same as that given in (1) with.

$$\sigma^2 = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_c^2 + \sigma_d^2 + \sigma_e^2, \quad \rho_5 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_c^2}{\sigma^2}$$

$$\rho_4 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_d^2}{\sigma^2}, \quad \rho_3 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2}{\sigma^2}, \quad \rho_2 = \frac{\sigma_a^2}{\sigma^2}, \quad \rho_1 = \frac{\sigma_b^2}{\sigma^2}$$

Now, let $Y = (Y_{1111}, \ldots, Y_{mdrc})$. Then $Y \sim N_{mdrc}(\theta 1_{mdrc}, \Sigma)$

This model is quite similar to the CRMM. We note that

$$\begin{array}{llll}
\sigma_a^2 = \sigma_b^2 \quad when \;\; d = m & iff \quad \rho_1 = \rho_2 & iff & \tau_1^2 = \tau_2^2 \\
\sigma_a^2 = \sigma_b^2 = 0 & iff \quad \rho_1 = \rho_2 = 0 & iff & \tau_1^2 = \tau_2^2 = \tau_3^2 \\
\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0 & iff \quad \rho_1 = \rho_2 = \rho_3 = 0 & iff & \tau_1^2 = \tau_2^2 = \tau_3^2 \\
\sigma_d^2 = 0 & iff \quad \rho_3 = \rho_4 & iff & \tau_5^2 = \tau_6^2 \\
\sigma_c^2 = \sigma_d^2 & iff \quad \rho_3 = \rho_4 = \rho_5 & iff & \tau_4^2 = \tau_5^2 = \tau_6^2 \\
\sigma_c^2 = 0 & iff \quad \rho_3 = \rho_5 & iff & \tau_4^2 = \tau_6^2
\end{array}$$

Therefore, to test $\sigma_a^2 = \sigma_b^2$, we test $\rho_1 = \rho_2$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2$, to test $\sigma_d^2 = 0$, we test $\rho_3 = \rho_4$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_5^2 = \tau_6^2$, similarly to test $\sigma_c^2 = 0$, we test $\rho_3 = \rho_5$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_4^2 = \tau_6^2$, to test $\sigma_a^2 = \sigma_b^2 = 0$, we test $\rho_1 = \rho_2 = 0$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2 = \tau_3^2$, then we use Bartlett's test, similarly, to test $\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0$, we test $\rho_1 = \rho_2 = \rho_3 = 0$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2 = \tau_3^2$, then we use Bartlett's test. To test $\sigma_c^2 = \sigma_d^2$, we test $\rho_3 = \rho_4 = \rho_5$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_4^2 = \tau_5^2 = \tau_6^2$, then we use Bartlett's test.

Application 5 In this application, we consider a balanced 4-way mixed effects model in which the third random effect nested in the interaction of the first two random effects and the fourth effect is fixed and interacts with the interaction of the first and second random effects. This model given by

$$Y_{ijk\ell} = \theta + a_i + b_j + (ab)_{ij} + c_{ijk} + \eta_\ell + (ab\eta)_{ij\ell} + e_{ijk\ell}$$

where θ, η_ℓ are unknown parameters such that $\sum_\ell \eta_\ell = 0$ and a_i, b_j, $(ab)_{ij}$, c_{ijk} $(ab\eta)_{ij\ell}$ and $e_{ijk\ell}$ are unobserved independent random variables such that

$a_i \sim N(0,\sigma_a^2)$, $b_j \sim N(0,\sigma_j^2)$, $(ab)_{ij} \sim N(0,\sigma_{ab}^2)$, $c_{ijk} \sim N(0,\sigma_c^2)$

$(ab\eta)_{ij\ell} \sim N(0,\sigma_{ab\eta}^2)$, $e_{ijk\ell} \sim N(0,\sigma_e^2)$

The parameter space for this model is given by

$$-\infty < \theta < \infty, \quad \sum_\ell \eta_\ell = 0, \quad \sigma_a^2 \geq 0, \quad \sigma_b^2 \geq 0, \quad \sigma_{ab}^2 \geq 0, \quad \sigma_c^2 \geq 0, \quad \sigma_{ab\eta}^2 \geq 0, \quad \sigma_e^2 > 0$$

We are interested in testing that $\sum_\ell \eta_\ell = 0$, $\sigma_a^2 = \sigma_b^2$, $\sigma_a^2 = \sigma_b^2 = 0$,

$\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0$, $\sigma_c^2 = 0$, $\sigma_{ab\eta}^2 = 0$ and $\sigma_c^2 = \sigma_{ab\eta}^2$. We note that the $Y_{ijk\ell}$ and $Y_{i'j'k'\ell'}$ are not independent for this model the $\text{cov}(Y_{ijk\ell}, Y_{i'j'k'\ell'})$ is the same as that given in (1).

$$\sigma^2 = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_c^2 + \sigma_{ab\eta}^2 + \sigma_e^2, \quad \rho_5 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_c^2}{\sigma^2}$$
$$\rho_4 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_{ab\eta}^2}{\sigma^2}, \quad \rho_3 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2}{\sigma^2}, \quad \rho_2 = \frac{\sigma_a^2}{\sigma^2}, \quad \rho_1 = \frac{\sigma_b^2}{\sigma^2}$$

Now, let $Y = (Y_{1111}, \ldots, Y_{mdrc})$. Then $Y \sim N_{mdrc}(\theta 1_{mdrc} + \eta \otimes 1_{mdr}, \Sigma)$

This model is quite similar to the CRMM. We note that

$$\begin{array}{llll}
\sigma_a^2 = \sigma_b^2 \quad \text{when } d = m & \text{iff} & \rho_1 = \rho_2 & \text{iff} \quad \tau_1^2 = \tau_2^2 \\
\sigma_a^2 = \sigma_b^2 = 0 & \text{iff} & \rho_1 = \rho_2 = 0 & \text{iff} \quad \tau_1^2 = \tau_2^2 = \tau_3^2 \\
\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0 & \text{iff} & \rho_1 = \rho_2 = \rho_3 = 0 & \text{iff} \quad \tau_1^2 = \tau_2^2 = \tau_3^2 \\
\sigma_{ab\eta}^2 = 0 & \text{iff} & \rho_3 = \rho_4 & \text{iff} \quad \tau_5^2 = \tau_6^2 \\
\sigma_c^2 = \sigma_{ab\eta}^2 & \text{iff} & \rho_3 = \rho_4 = \rho_5 & \text{iff} \quad \tau_4^2 = \tau_5^2 = \tau_6^2 \\
\sigma_c^2 = 0 & \text{iff} & \rho_3 = \rho_5 & \text{iff} \quad \tau_4^2 = \tau_6^2
\end{array}$$

If we want to test that the $\sum_\ell \eta_\ell = 0$, we just follow the same procedure given in the first three examples,

Therefore, to test $\sigma_a^2 = \sigma_b^2$, we test $\rho_1 = \rho_2$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2$, to test $\sigma_a^2 = \sigma_b^2 = 0$, we test $\rho_1 = \rho_2 = 0$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2 = \tau_3^2$, to test $\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0$, we test $\rho_1 = \rho_2 = \rho_3 = 0$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2 = \tau_3^2$, then we use Bartlett's test, then we use Bartlett's test, to test $\sigma_{ab\eta}^2 = 0$, we test $\rho_3 = \rho_4$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_5^2 = \tau_6^2$, to test $\sigma_c^2 = \sigma_{ab\eta}^2$, we test $\rho_3 = \rho_4 = \rho_5$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_4^2 = \tau_5^2 = \tau_6^2$, then we use Bartlett's test, to test $\sigma_c^2 = 0$, we test $\rho_3 = \rho_5$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_4^2 = \tau_6^2$.

Application 6. In this application, we consider a balanced 4 -way mixed effects model which is given by

$$Y_{ijk\ell} = \theta + a_i + b_j + (ab)_{ij} + \gamma_k + \eta_\ell + (ab\gamma)_{ijk} + (ab\eta)_{ij\ell} + e_{ijk\ell}$$

where $\theta,\ \eta_\ell$, γ_k are unknown parameters such that $\sum_k \gamma_k = 0$, $\sum_\ell \eta_\ell = 0$ and $a_i,\ b_j,\ (ab)_{ij},\ (ab\gamma)_{ijk},\ (ab\eta)_{ij\ell}$ and $e_{ijk\ell}$ are unobserved independent random variables such that

$$a_i \sim N(0,\sigma_a^2),\quad b_j \sim N(0,\sigma_j^2),\quad (ab)_{ij} \sim N(0,\sigma_{ab}^2),\quad (ab\gamma)_{ijk} \sim N(0,\sigma_{ab\gamma}^2)$$

$$(ab\eta)_{ij\ell} \sim N(0,\sigma_{ab\eta}^2),\quad e_{ijk\ell} \sim N(0,\sigma_e^2)$$

The parameter space for this model is given by

$$-\infty < \theta < \infty,\quad \sum_\ell \eta_\ell = 0,\quad \sum_k \gamma_k = 0,\quad \sigma_a^2 \ge 0,\quad \sigma_b^2 \ge 0$$

$$\sigma_{ab}^2 \ge 0,\quad \sigma_{ab\gamma}^2 \ge 0,\quad \sigma_{ab\eta}^2 \ge 0,\quad \sigma_e^2 > 0$$

We are interested in testing that $\sum_\ell \eta_\ell = 0$, $\sum_k \gamma_k = 0$, $\sigma_a^2 = \sigma_b^2$, $\sigma_a^2 = \sigma_b^2 = 0$, $\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0$, $\sigma_{ab\gamma c}^2 = 0$, $\sigma_{ab\eta}^2 = 0$ and $\sigma_{ab\gamma}^2 = \sigma_{ab\eta}^2$. We note that the $Y_{ijk\ell}$ and $Y_{i'j'k'\ell'}$ are not independent for this model the $\operatorname{cov}(Y_{ijkl}, Y_{i'j'k'\ell'})$ is the same as that given in (1).

$$\sigma^2 = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_{ab\gamma}^2 + \sigma_{ab\eta}^2 + \sigma_e^2,\quad \rho_5 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_{ab\gamma}^2}{\sigma^2}$$

$$\rho_4 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_{ab\eta}^2}{\sigma^2},\quad \rho_3 = \frac{\sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2}{\sigma^2},\quad \rho_2 = \frac{\sigma_a^2}{\sigma^2},\quad \rho_1 = \frac{\sigma_b^2}{\sigma^2}$$

Now, let $Y = (Y_{1111}, \ldots, Y_{mdrc})$. Then

$$Y \sim N_{mdrc}(\theta 1_{mdrc} + 1_c \otimes \gamma \otimes 1_{md} + \eta \otimes 1_{mdr}, \Sigma)$$

This model is quite similar to the CRMM. We note that

$$
\begin{array}{llll}
\sigma_a^2 = \sigma_b^2 \quad when \;\; d = m & iff \quad \rho_1 = \rho_2 & iff & \tau_1^2 = \tau_2^2 \\
\sigma_a^2 = \sigma_b^2 = 0 & iff \quad \rho_1 = \rho_2 = 0 & iff & \tau_1^2 = \tau_2^2 = \tau_3^2 \\
\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0 & iff \quad \rho_1 = \rho_2 = \rho_3 = 0 & iff & \tau_1^2 = \tau_2^2 = \tau_3^2 \\
\sigma_{ab\eta}^2 = 0 & iff \quad \rho_3 = \rho_4 & iff & \tau_5^2 = \tau_6^2 \\
\sigma_{ab\gamma}^2 = \sigma_{ab\eta}^2 & iff \quad \rho_3 = \rho_4 = \rho_5 & iff & \tau_4^2 = \tau_5^2 = \tau_6^2 \\
\sigma_{ab\gamma}^2 = 0 & iff \quad \rho_3 = \rho_5 & iff & \tau_4^2 = \tau_6^2
\end{array}
$$

If we want to test that the $\sum_{\ell} \eta_\ell = 0$, that the $\sum_k \gamma_k = 0$, we just follow the same procedure given in the first three examples, Therefore, to test $\sigma_a^2 = \sigma_b^2$, we test $\rho_1 = \rho_2$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2$, to test $\sigma_a^2 = \sigma_b^2 = 0$, we test $\rho_1 = \rho_2 = 0$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2 = \tau_3^2$, to test $\sigma_a^2 = \sigma_b^2 = \sigma_{ab}^2 = 0$, we test $\rho_1 = \rho_2 = \rho_3 = 0$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_1^2 = \tau_2^2 = \tau_3^2$, then we use Bartlett's test, then we use Bartlett's test, to test $\sigma_{ab\eta}^2 = 0$, we test $\rho_3 = \rho_4$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_5^2 = \tau_6^2$, to test $\sigma_{ab\gamma}^2 = \sigma_{ab\eta}^2$, we test $\rho_3 = \rho_4 = \rho_5$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_4^2 = \tau_5^2 = \tau_6^2$, then we use Bartlett's test, to test $\sigma_{ab\gamma}^2 = 0$, we test $\rho_3 = \rho_5$ for the CRMM and in the transformed model it is equivalent for testing that $\tau_4^2 = \tau_6^2$.

5. Discussion and Conclusion

The approach in this paper permits us to find procedures for any different mixed models simultaneously because of the wider assumption we made about the means. Our results can directly be extended to the cases when any numbers of fixed effects are added to the mixed and random effects models given in the example 5-6, as long as the added fixed effects do not interact with any random effects. The reason for the existence of optimal procedures in our approach is that the model can be transformed in to a product of models (because the correlation coefficients can be negative as long as the covariance matrix is positive definite). However, in the mixed models (see example 5-6) the correlation coefficients must be non-negative. In this case the transformed model is not a product of models. One more advantage in our approach is that it is possible to get all the required formulas in terms of the original variables $Y_{ijk\ell}$ and we do not need to transform to variables x_i for computing statistics discussion in this paper. We write below the expression of the various formulas given in equation (9) in term of $Y_{ijk\ell}$

$$SS_1 = df_1 M_1 = drc\sum_i (\bar{Y}_{i...} - \bar{Y}_{....} - \hat{\beta}_1^{(i)})^2$$
$$SS_2 = df_2 M_2 = mrc\sum_j (\bar{Y}_{.j..} - \bar{Y}_{....} - \hat{\beta}_2^{(j)})^2$$
$$SS_3 = df_3 M_3 = rc\sum_i \sum_j (\bar{Y}_{ij..} - \bar{Y}_{.j..} - \bar{Y}_{i...} + \bar{Y}_{....} - \hat{\beta}_3^{(ij)})^2$$
$$SS_4 = df_4 M_4 = c\sum_i \sum_j \sum_k (\bar{Y}_{ijk.} - \bar{Y}_{ij..} - \hat{\beta}_4^{(ijk)})^2$$
$$SS_5 = df_5 M_5 = r\sum_i \sum_j \sum_\ell (\bar{Y}_{ij.\ell} - \bar{Y}_{ij..} - \hat{\beta}_5^{(ij\ell)})^2$$
$$SS_6 = df_6 M_6 = \sum_i \sum_j \sum_k \sum_\ell (\bar{Y}_{ijk\ell} - \bar{Y}_{ij.\ell} - \bar{Y}_{ijk.} + \bar{Y}_{ij..} - \hat{\beta}_6^{(ijk\ell)}$$

Where $\hat{\beta}_h$ is the OLS estimators of $\hat{\beta}_h$, $h = 1, \ldots, 6$ in the OLM that occurs when in the OLM that occures when $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = 0$, in the CRMM. In order to get the remaining $(\hat{\mu}, M_1, M_2, M_3, M_4, M_5, M_6)$ in terms of $Y_{ijk\ell}$. We define

$$W_1 = \{\mu : \sum_{h=0}^{6} H_h \beta_h,\ \beta_s \in T_s,\ \ s \neq 1,\ \ \beta_1 \in Q_1\}$$

$$W_h = \{\mu = \sum_{h=0}^{6} H_s \beta_s,\ \beta_s \in T_s,\ s \neq h,\ \ \beta_h \in Q_h\},\quad h = 2, \ldots, 6$$

Where

$$H_0 = 1_{mdrc},\ H_1 = 1_{drc} \otimes I_m,\ H_2 = 1_{cr} \otimes I_d \otimes I_m,\ H_3 = 1_{cr} \otimes I_{md}$$

$$H_4 = 1_c \otimes I_{mdr},\ H_5 = I_c \otimes 1_r \otimes I_{md},\ H_6 = I_{mdrc}$$

And

$$\left\|P_{V|W_1}\mu\right\|^2 = \left\|P_{T_1^*|Q^*}\gamma_1\right\|^2 \quad , \left\|P_{V|W_1}Y_1\right\|^2 = \left\|P_{T_1^*|Q_1^*}Y_1^*\right\|^2 \qquad \textbf{[10]}$$

$$\dim(V \mid W_1) = \dim(T_1^* \mid Q_1^*) \qquad \textbf{[11]}$$

Finally, we not note from equation (9-11) that for all F_i, $i = 1, 2, \cdots, 6$, the numerator sum of square $\left\|P_{V|W_i}Y_i\right\|^2$ and degree of freedom $t_i - q_i$ for i $for\ i = 1, 2, \cdots, 6$ are the same as for the OLM. Therefor, to find the approparate F-statistic for the CRMM for a type i hypothesis, we merely take the F -statistic for the OLM and replace σ^2 and dfe with M_i and df_i depending on whether the hypothesis is of type i $for\ i = 1, 2, \cdots, 6$, similarly, to find the non-centrality parameter, we merely take the non-centrality parameter in the OLM and replace σ^2 with τ^2 depending on the hypothesis is of type $i \;\; for \; i = 1,2,\cdots,6$. We would like to point out that it has not been possible to find an exact F-test for testing that $\sigma_a^2 = 0$ as it is clear from examples 5-6 but it is worth trying.

References

Al-Sakkal, G.A. (1999). Optimal Procedures for Certain Crossed Repeated Measures Model. M.Sc. Thesis, Al-Mustansiriyah, University, Iraq.

Arnold, S. F. (1973). Application of the theory of Products of Problems to certain Patterned covariance matrices. *Ann. Statist.* 1,682-699.

Arnold, S. F. (1979). A Coordinate-Free Approach to Finding Optimal Procedures for Repeated Measures Designs. *Ann. Statist.* 7,812-822.

Baayen, R. H., Davidson D. J., & Bates D.M. (2008). Mixed-effects modeling with crossed random effects for subjects and items. *Journal of Memory and Language*, 59(4), 390-412.

Cox, G. M., & Cochran W. G. (1992). *Experimental designs*. New York: Wiley and Sons.

Gabbara, S. D. (1985). Nested and Crossed Repeated Measures Models. Unpublished, Ph. D. Thesis, The Pennsylvania State University, USA.

Hoshmand R. (2006). *Design of experiments for agriculture and natural sciences*. 2nd Edition, Taylor and Francis Group.

Lehman, E. L. (1959). *Testing statistical hypotheses*. New York: Wiley.

Rhonda, D., Szczesniak, D. L., & Raouf S. A. (2016). Semi parametric mixed models for nested repeated measures applied to ambulatory blood pressure monitoring data. *The Journal of Modern Applied Statistical Methods*, 15(1), 255-275.

Saarinen, F. (2004). Using mixed models in a cross-over study with repeated measurements within periods. Thesis submitted to the Dept. of Mathematical Statistics, Stockholm University, SE-106 91 Stockholm, Sweden.