Analysis of Elastic Beams on Linear and Nonlinear Foundations Using Finite Difference Method

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Abstract: An approximate method is developed to analyze the deflection in beams and beam-column by solving the differential equation for the elastic deformation of beam and beam-column. The analysis is performed using the central difference of finite difference method for the Euler-Bernoulli beam and beam-column supported on an elastic, nonlinear foundation with rigid or elastic discrete supports. To make a verification of the results, Laplace Transformation method was used to solve the elastic differential equation of beam and beam-column based on linear elastic supports and the results were compared with the finite difference method. Two types of beams were selected, simply supported and fixed-fixed with five elastic supports of an idealized soil. In the nonlinear idealization, the division of force into many levels were assumed and based on these forces, the equivalent displacements were obtained from an assumed power law equation by using the finite difference method. Central finite difference scheme, which has a second order, was used throughout the numerical analysis with five nonlinear behavior of springs separated by an equal distance between them.

Keywords: Finite Difference Method, Euler-Bernoulli Beam, Laplace Transformation

1. Introduction

Beams on an elastic foundation have been solved by many researchers and analytical solutions of the differential equation have been proposed (Cook, 2007; Miyahara & Ergatoudis, 1976). The geometric stiffness matrix was formulated and derived for beams on elastic foundation by Eisenber et al. (1986). Many authors used a finite element technique to find an approximate solution. Two-parameter elastic foundations were formulated to analyze beams based on exact displacement function (Zhaohua & Cook, 1983). Analysis of finite element beam column on elastic Winkler foundation was carried out using exact stiffness matrix terms (Yankelevsky & Eisenberger, 1986). Lower order of finite strip method developed for the analysis of soil-iteration models. At the early stage, the model was used for soil layered under vertical load with uniform soil properties. The change of soil properties in the longitudinal direction was included in the research by Cheung et al. (1985) and Oskoorouchi et al. (1991). Mixing between finite strip method and soil spring system has been developed and applied to study the plate vibration responses on elastic foundation with different boundary conditions (Huang et al., 2001; Chow et al., 1989). Vallabhan and Das (1988) estimated a non-dimensional third parameter using iterative procedure to represent the distribution displacement of beams rested on elastic foundation. Omurtag et al. (1997) used a mixed-type formulation based on Gateaux differential for the derivation of Kirchhoff plate-elastic foundation interaction. Binesh (2012) used a mesh-free method for the analysis of a beam on two parameter elastic foundation. Sato et al. (2008) obtained an exact solution for beam on elastic foundation in static and free vibration problems based on equidistant elastic supports. Jumel et al. (2011) proposed a first order correction to take into account of interface elasticity and transverse anticlastic curvature of flexible substrate. Borak and Marcian (2014) used modified Bettis theorem to develop an alternative analytical solution of beams on an elastic foundation. The calculation was based on the determination of beam's deflection on an elastic foundation from the deflection of a reference beam which is topologically equivalent. In this paper, the nonlinear assumption of soil behavior is used to analyze Euler-Bernoulli elastic beam and beam-column under compression load rested on it. Numerical method based on the finite difference method is used for the analysis of fourth-order linear ordinary differential equation of beam and beam-column. The constants in this equation are determined by using the boundary conditions of simply supported and fixed-fixed ends. In addition, closed form solution based on the Laplace transformation method is used to get the results under the same conditions. Numerical examples are illustrated for both elastic beam and beam-column on the nonlinear soil behavior and the results are showed in figures and tables.

2. Modelling of Soil Mechanical Properties and Algorithm

Throughout analysis of any foundation, soil is not a linear material and for modeling it as a linear material could cause a considerable error. Soil actually behaves as a hyperbolic curve in relationship of stress-strain (Eisenberger et al., 1986). Load deflection relationship curve might be assumed to exist, as shown in Figures 1-5 when modeling the reaction of a soil foundation. In every iteration of the force level, the tangent of the curve is obtained which means the slope of the force-displacement curve of soil. In this study, this curve is obeyed according to the nonlinear behavior of soil that defined in Eq. (1)

$$P = 0.5 P_u \varepsilon^n \tag{1}$$

where P_u is the ultimate soil bearing capacity of the soil in kN/m^2 and ε is the strain of soil underneath the footing. n is the power law variation that changes with the applied force and displacement. Here, the strain equals to the actual displacement because the length is assumed for one meter. Five nodes are idealized for the nonlinear soil reaction using the finite difference method to find the displacements. The load per unit area of foundation is plotted with displacement for each spring as shown in Figures 1-5. For each figure, the corresponding deflection is selected based on five loads chosen as 0.1, 0.25, 0.5, 0.75, and 1.0. These curves behavior which represented the soil mechanical activities are working based on Eq. (1). The value of power n in Eq. (1) goes to decline with increasing of load step from 0.1 to 1.0. The differential equation that describes the elastic deflection curve of Euler-Bernoulli beam-column on a nonlinear elastic foundation under the action of a distributed load is governed by Eq. (2)

$$\frac{d^2}{dx^2}\left[EI\frac{d^2y}{dx^2}\right] + P_\alpha \frac{d^2y}{dx^2} + k(x)y = w(x) \tag{2}$$

where $y(x)$ is the deflection of the beam, E is the modulus of elasticity of the material made for the beam, I is the moment of inertia of the cross-section, P_α is the axial load applied at the ends to the beam, $w(x)$ is the applied distributed load and $k(x)$ is the foundation modulus. In the linear analysis of foundation, $k(x)$ is taken as a constant number while in the nonlinear foundation analysis varies in its values with respect to its position.

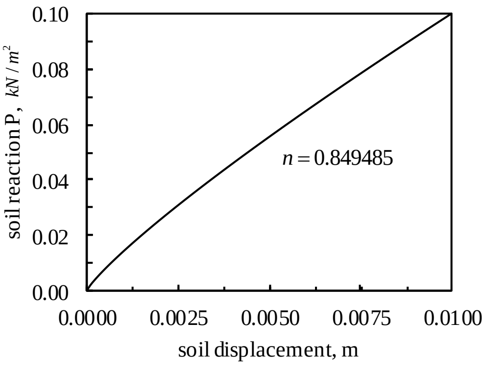

Figure 1: Load-displacement curve of soil at $P = 0.10\, kN / m^2$

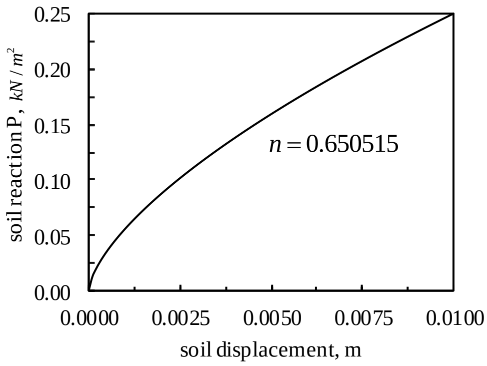

Figure 2: Load-displacement curve of soil at $P = 0.25\, kN / m^2$

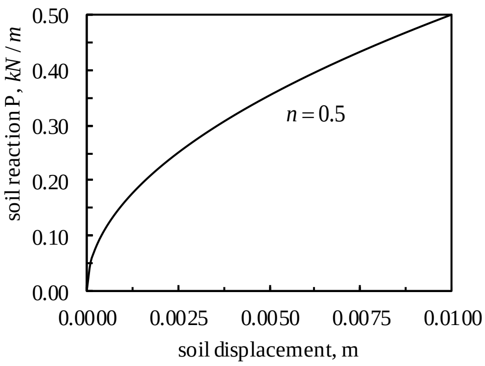

Figure 3: Load-displacement curve of soil at $P = 0.50\, kN / m^2$

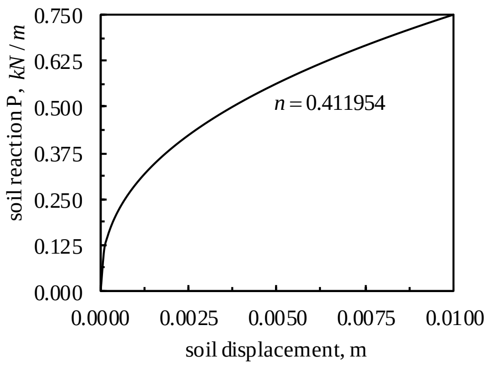

Figure 4: Load-displacement curve of soil at $P = 0.75\, kN / m^2$

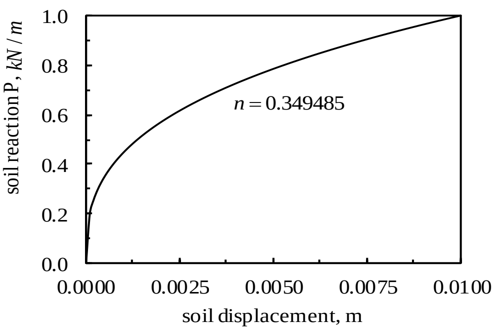

Figure 5: Load-displacement curve of soil at $P = 1.0\, kN / m^2$

With applying finite difference scheme, Eq. (2) can be written for the elastic beam-column on nonlinear foundation as

$$y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2} + \frac{P_\alpha}{EI}(y_{i+1} - 2y_i + y_{i-1})h^2 = \frac{w_i h^4}{EI} - \frac{k_i y_i h^4}{EI} \tag{3}$$

In the case of elastic beam on nonlinear foundation can be calculated in the same Eq. (3) with $P_\alpha = 0$. The second and fourth order derivatives in Eq. (2) are substituted by second order central difference approximation in the finite difference equation (Eq. (3)). The finite difference grid is used for the analysis of beam and beam-column rested on elastic nonlinear foundation with taking a limit number of idealized springs. The distances between the springs are taken an equal and the axial load as a compression load at the ends. In the conventional finite difference analysis, the geometric and boundary conditions of the equilibrium differential equation are considered. The condition that y' and y'' are zero at station i is approximated using Eq. (4) and Eq. (5) respectively, as

$$y_{i+1} - y_{i-1} = 0 \tag{4}$$

$$y_{i-1} - 2y_i + y_{i+1} = 0 \tag{5}$$

Figure 6 shows the simply supported beam-column on five nonlinear idealized springs with equal distances between them.

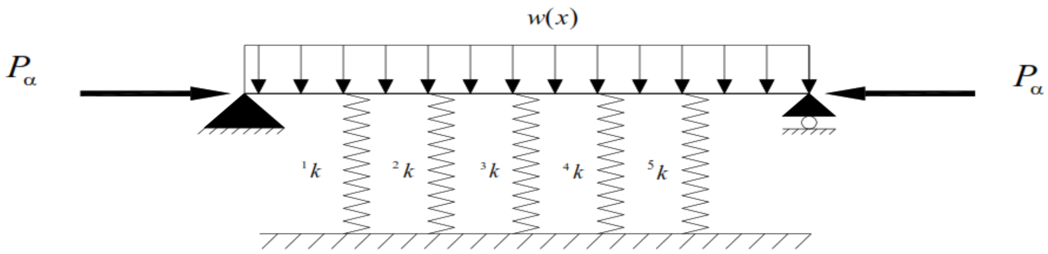

Figure 6: Simply supported beam-column rested on five springs

The algorithm processes in this study are followed as the following steps: In the first load step ($P = 0.1$) with the constant last strain ($\varepsilon = 0.01$), the power n is determined to equal 0.849485 according to the Eq. (1) with taking $P_u = 10.0$. After then, the stiffness (k) is equal to the slope of the load-deflection curve by assuming the deflection equals to the strain for idealized springs. Then the k values for the five springs are approximately linear for this step and equal to 10.0 because the exact values in this step can't be determined without presenting the springs deflection. By applying the finite difference technique (Eq. (3)), values of the springs deflection can be determined based on the constant value of stiffness ($k = 10.0$) for all assumed nodes. For the second step ($P = 0.25$), the value of power n can be determined with constant final strain ($\varepsilon = 0.01$). In this step, with determining the values of deflections from the first step, the new stiffness (k) for all five nodes can be calculated by substituting the deflection values in the derivative of Eq. (1) ($4.24743/\varepsilon^{0.150515}$) which considers the slope of the load-deflection curve then the corrected deflections can be evaluated by applying the same equation of finite difference method. For the third step ($P = 0.5$), the same procedures are applied from calculating the power n and evaluating the stiffness based on the second step deflections to reach the new corrected deflections. The processes are repeated for the fourth step ($P = 0.75$) and fifth step ($P = 1.0$). The observation of this algorithm is the load-deflection curve goes from linear at the first step to nonlinear behavior ending with the power $n = 0.349485$. The soil beneath the beam or beam-column behaves actually according to the mentioned five steps. The reason to stop of these processes at the last step is the small difference between the values of numerical method (finite difference method) and the exact solution (Laplace Transformation).

2.1 Laplace Transformation Method

In this section, the Laplace transformation method is used to find the deflections of beam rested on linear elastic foundation. To apply this method, the basic principle is defined as $F(s)$ be a given function. The Laplace transform $F(s)$ of function $f(t)$ is defined by

$$\mathrm{L}\left\{f(x)\right\} = \int_0^\infty e^{-sx} f(x)dx,\ s > 0 \tag{6}$$

By rearranging the fourth order differential equation in Eq. (2), the equation can be written as

$$y^{(4)}(x) + 4\beta^4 y(x) + \alpha^2 y''(x) = \frac{w(x)}{EI} \tag{7}$$

where, $4\beta^4 = k / EI$ (k is the foundation modulus) and $\alpha^2 = P_\alpha / EI$ (P_α is the axial applied load at the beam ends). By taking the Laplace transformation for the fourth order differential equation in Eq. (7) under constant value of the distributed load $w(x)$, the equation becomes

$$f(s) = \frac{EIs\left(\left(\alpha^2 + s^2\right)\left(sy(0) + y'(0)\right) + s\,y''(0) + y'''(0)\right) + w}{(EIs)\left(4\beta^4 + s^4 + \alpha^2 s^2\right)} \tag{8}$$

For simply supported beam rested on elastic foundation under axial load at the ends of the beam, the boundary conditions at the ends which satisfying the Eq. (8) are deflection and moment equal to zero ($y(0) = y(l) = y''(0) = y''(l) = 0$) and Eq. (8) can be written as

$$f(s) = \frac{w + EIs\left(c_2 + c_1\left(s^2 + \alpha^2\right)\right)}{EIs\left(s^4 + s^2\alpha^2 + 4\beta^4\right)} \tag{9}$$

where $c_1 = y'(0)$ and $c_2 = y''(0)$. In addition, by applying the boundary conditions for the fixed-fixed beam with axial load at the ends with taking the deflection and slope at the ends equal to zero ($y(0) = y(l) = y'(0) = y'(l) = 0$), Eq. (8) can be written as

$$f(s) = \frac{w + EIs\left(c_2 + c_1 s\right)}{EIs\left(s^4 + s^2\alpha^2 + 4\beta^4\right)} \tag{10}$$

where $c_1 = y''(0)$ and $c_2 = y'''(0)$.

3. Numerical Examples

The aim of this section is to validate the solution procedure for the determination of the beam deflection rested on elastic foundation. To do so, several examples of the analysis of beam columns are illustrated based on the analytical and numerical solutions. The first example explains beam ($\alpha = 0.0$) and beam-column ($\alpha = 1.0$) with simply supported on elastic foundations while the second one is treating with beam and beam-column with fixed ends. For both examples, the parameters are selected as a following data: $k = 1.0$, $k = 2.0$, $EI = 1.0$, $P_u = 10.0$ and $L = 1.0$.

The constants for simply supported and fixed-fixed respectively are $c_1 = 0.0412493$, $c_2 = 0.0412493$ and $c_1 = 0.0831848$, $c_2 = -0.499307$. These constants are used for Laplace transformation method in Eqs. (9) and (10). The results in the Figures 7-14 show deflection by using analytical method for both beam and beam-column in the case of simply supported and fixed-fixed conditions. Tables 1 and 2 show the numerical deflection of beam and beam-column on elastic foundation for both simply supported and fixed-fixed, respectively based on central difference of finite difference method. Tables 3 and 4 represent stiffness values and corresponding deflection which should be used in the tangent equation at each level for simply supported beam. In addition, Tables 5 and 6 show the stiffness values and corresponding deflection for fixed-fixed beam rested on nonlinear foundation, respectively.

Table 1: Numerical deflection of simply supported beam and beam-column on elastic foundation (m).

Divided length, m	Deflection $k = 1, \alpha = 0$	Deflection $k = 2, \alpha = 0$	Deflection $k = 1, \alpha = 1$	Deflection $k = 2, \alpha = 1$
0.0	0.000000	0.000000	0.000000	0.000000
1/6	0.006680	0.006611	0.007437	0.007351
2/6	0.011451	0.011330	0.012760	0.012611
3/6	0.013168	0.013029	0.014679	0.014506
4/6	0.011451	0.011330	0.012760	0.012611
5/6	0.006680	0.006611	0.007437	0.007351
6/6	0.000000	0.000000	0.000000	0.000000

Table 2: Numerical deflection of fixed-fixed beam and beam-column on elastic foundation (m).

Divided length, m	Deflection $k = 1, \alpha = 0$	Deflection $k = 2, \alpha = 0$	Deflection $k = 1, \alpha = 1$	Deflection $k = 2, \alpha = 1$
0.0	0.000000	0.000000	0.000000	0.000000
1/6	0.001123	0.001120	0.001148	0.001145
2/6	0.002566	0.002559	0.002634	0.002627
3/6	0.003175	0.003167	0.003263	0.003255
4/6	0.002566	0.002559	0.002634	0.002627
5/6	0.001123	0.001120	0.001148	0.001145
6/6	0.000000	0.000000	0.000000	0.000000

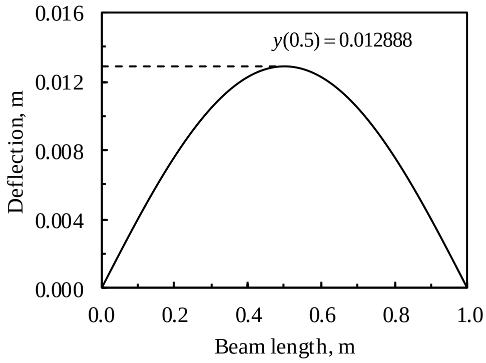

Figure 7: Analytical deflection of simply supported beam on linear elastic foundation with $k = 1$ and $\alpha = 0$.

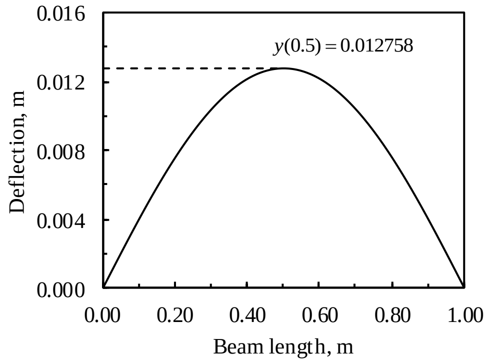

Figure 8: Analytical deflection of simply supported beam on linear elastic foundation with $k = 2$ and $\alpha = 0$.

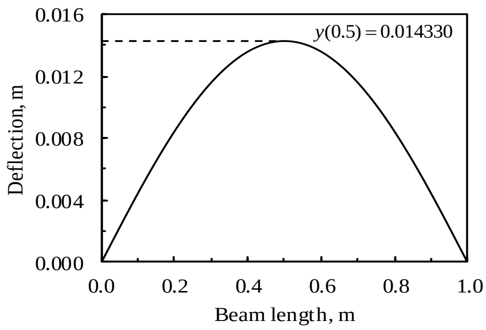

Figure 9: Analytical deflection of simply supported beam-column on linear elastic foundation with $k = 1$ and $\alpha = 1$.

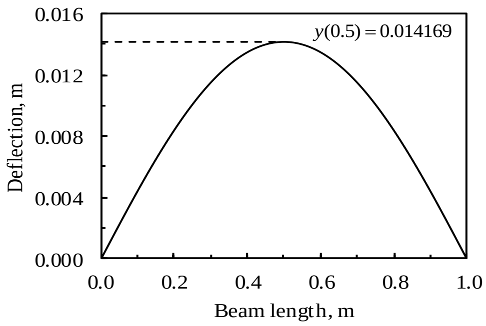

Figure 10: Analytical deflection of simply supported beam-column on linear elastic foundation with $k = 2$ and $\alpha = 1$.

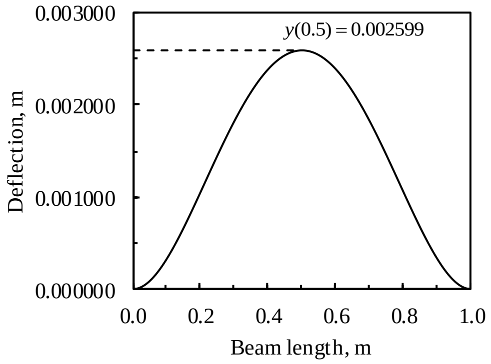

Figure 11: Analytical deflection of fixed-fixed beam on linear elastic foundation with $k = 1$ and $\alpha = 0$.

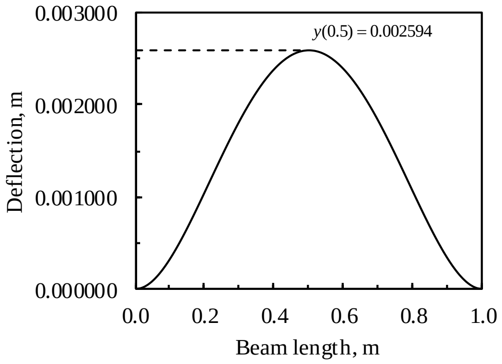

Figure 12: Analytical deflection of fixed-fixed beam on linear elastic foundation with $k = 2$ and $\alpha = 0$.

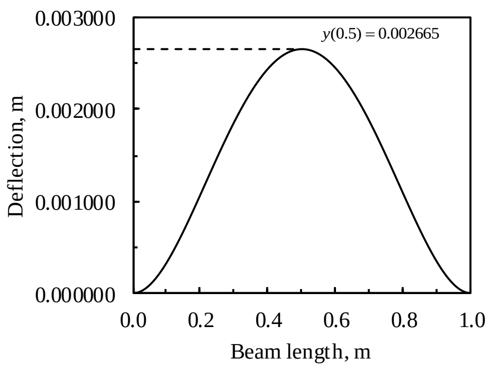

Figure 13: Analytical deflection of fixed-fixed beam on linear elastic foundation with $k=1$ and $\alpha=1$.

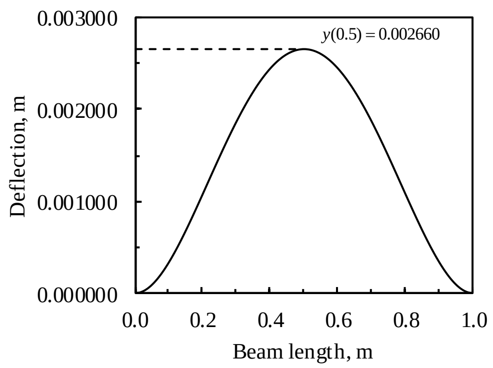

Figure 14: Analytical deflection of fixed-fixed beam on linear elastic foundation with $k=2$ and $\alpha=1$.

Table 3: Stiffness of load per unit area and deflection curve for simply supported beam (kN/m^2).

Beam length, m	${}^{1}k$	${}^{2}k$	${}^{3}k$	${}^{4}k$	${}^{5}k$
0.10	10.0000	43.2154	71.7068	60.8176	70.5617
0.25	10.0000	35.8092	54.8407	44.3825	49.7651
0.5	10.0000	34.1072	51.1660	40.9110	45.4655
0.75	10.0000	35.8092	54.8407	44.3825	49.7651
1.00	10.0000	43.2154	71.7068	60.8176	70.5617

Table 4: Deflection of simply-supported beam rested on nonlinear foundation for five iterations (m).

Beam length, m	${}^{1}u$	${}^{2}u$	${}^{3}u$	${}^{4}u$	${}^{5}u$
0.10	0.000610	0.001216	0.003161	0.003396	0.004345
0.25	0.001045	0.002078	0.005401	0.005808	0.007432
0.5	0.001201	0.002387	0.006204	0.006674	0.008540
0.75	0.001045	0.002078	0.005401	0.005808	0.007432
1.00	0.000610	0.001216	0.003161	0.003396	0.004345

Table 5: Stiffness of load per unit area and deflection curve for fixed-fixed beam (kN/m^2).

Beam length, m	${}^{1}k$	${}^{2}k$	${}^{3}k$	${}^{4}k$	${}^{5}k$
0.10	10.0000	16.7481	57.3219	110.6000	145.3530
0.25	10.0000	14.7920	42.9680	73.3240	89.7250
0.5	10.0000	14.3261	39.8927	65.9587	79.2439
0.75	10.0000	14.7920	42.9680	73.3240	89.7250
1.00	10.0000	16.7481	57.3219	110.6000	145.3530

Table 6: Deflection of fixed-fixed beam rested on nonlinear foundation for five iterations (m).

Beam length, m	${}^{1}u$	${}^{2}u$	${}^{3}u$	${}^{4}u$	${}^{5}u$
0.10	0.000110	0.000272	0.000511	0.000718	0.000925
0.25	0.000251	0.000620	0.001162	0.001632	0.002102
0.5	0.000310	0.000767	0.001437	0.002016	0.002596
0.75	0.000251	0.000620	0.001162	0.001632	0.002102
1.00	0.000110	0.000272	0.000511	0.000718	0.000925

4. Conclusion

In this study, a finite difference method for the analysis of beam and beam-column resting on nonlinear elastic foundation is formulated based on the iterative procedure. In addition, to control the accuracy of the numerical method, exact solution for beam and beam-column of Euler-Bernoulli on a nonlinear elastic foundation is investigated based on the proposed formula in Eq. (1). The nonlinear load-displacement curve of the soil is plotted at each of the load step with determining the power n at each of load level. With applying the finite difference procedures at each load level, the displacements corresponding to it are determined then used in Eq. (3) to find the next level displacements under the equal of point load at each of the five interior nodes. The iterative procedure converges rapidly to the solution after the fifth trail because the value of the power n decreases with the advance steps and the curve approaches to the ideal curve path of soil. The final strain of the soil behavior at all levels is selected as a constant value ($\varepsilon = 0.01$) and the value of the power (n) is determined at each load level. Accuracy is controlled by using the Laplace Transformation method with the inverse Eq. (10) which refers to the closed form solution. Two examples are solved for both beam and beam-column based on Euler-Bernoulli theory which show the characteristic features for applying the both methods. The analytical method using Laplace transformation is idealized in the figures with evaluating maximum displacement for all cases. It is shown that, for a twice stiffness value of beam and beam-column, the foundation deflections are reduced for simply supported and fixed-fixed ends. On the other hand, the deflections increase in the case of beam-column which refers to presence of axial load ($\alpha = 1$) than in the case of beam only ($\alpha = 0$). Moreover, the deflection of foundation rested on nonlinear soil behavior and the effect of compression axial with simply supported ends is greater than that for the fixed-fixed ends.

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References

Binesh, S. M. (2012). Analysis of beam on elastic foundation using the radial point interpolation method. *Scientia Iranica*, 19(3), 403-409.

Borák, L., & Marcián, P. (2014). Beams on elastic foundation using modified Betti′ s theorem. *International Journal of Mechanical Sciences*, 88,17-24.

Cheung, Y. K., Tham, L. G., & Guo, D. J. (1985). Applications of finite strip and layer methods in micro-computers. In International Conference on Numerical Methods in Geomechanics (ICONMIG).

Chow, Y. K., Swaddiwudhipong, S., & Phoon, K. F. (1989). Finite strip analysis of strip footings: Horizontal loading. *Computers and Geotechnics*, 8 (1), 65-86.

Cook, R. (2007). *Concepts and Applications of Finite Element Analysis*. John Wiley & Sons.

Eisenberger, M., Yankelevsky, D. Z., & Clastornik, J. (1986). Stability of beams on elastic foundation. *Computers & Structures*, 24(1), 135-139.

Huang, M. H., & Thambiratnam, D. P. (2001). Analysis of plate resting on elastic supports and elastic foundation by finite strip method. Computers & Structures, 79(29), 2547-2557.

Jumel, J., Budzik, M. K., & Shanahan, M. E. (2011). Beam on elastic foundation with anticlastic curvature: Application to analysis of mode I fracture tests. *Engineering Fracture Mechanics,* 78(18), 3253-3269.

Miyahara, F., & Ergatoudis, J. G. (1976). Matrix analysis of structure-foundation interaction. *Journal of the Structural Division*, 102(1), 251-265.

Omurtag, M. H., Özütok, A., Aköz, A. Y., & Ozcelik Y. (1997). Free vibration analysis of Kirchhoff plates resting on elastic foundation by mixed finite element formulation based on Gateaux differential. *International Journal for Numerical Methods in Engineering*, 40(2), 295-317.

Oskoorouchi, A. M., Novrouzian, B., De Roeck, G., & Van Den Broeck, J. (1991). Zoned finite strip method and its applications in geomechanics. *Computers and Geotechnics*, 11(4), 265-294.

Sato, M., Kanie, S., & Mikami, T. (2008). Mathematical analogy of a beam on elastic supports as a beam on elastic foundation. *Applied Mathematical Modelling*, 32(5), 688-699.

Vallabhan, C. G., & Das, Y. C. (1988). Parametric study of beams on elastic foundations. *Journal of Engineering Mechanics*, 114(12), 2072-2082.

Yankelevsky, D. Z., & Eisenberger, M. (1986). Analysis of a beam column on elastic foundation. *Computers & Structures*, 23(3), 351-356.

Zhaohua, F., & Cook, R. D. (1983). Beam elements on two-parameter elastic foundations. *Journal of Engineering Mechanics,* 109(6), 1390-1402.