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DEGREE REDUCTION OF BÉZIER CURVES WITH CHEBYSHEV WEIGHTED G^3-CONTINUITY

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Abstract. This paper considers Chebyshev weighted G^3-multi-degree reduction of Bézier curves. Exact degree reduction is not possible, based on this fact, approximative process to reduce a given Bézier curve of higher degree n to a Bézier curve of lower degree m, $m < n$ is required. The weight function $w[t] = 2t(1-t)$, $t \in [0,1]$ is used with the L_2 -norm in multi degree reduction with G^3- continuity at the end points of the curve. Explicit results and comparisons are verified by examples. The numerical result obtained from the new method yields minimum approximation error, improves the approximation in the middle of the curve, and shows up helpful applications to many scientists and engineers on how to design and reconstruct complex systems.

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1. INTRODUCTION

In degree reduction, we approximate a Bézier curve of degree n by a Bézier curve of degree $m, m < n$ under the satisfaction of boundary conditions and minimum error conditions. The struggles of finding a solution are disturbed by the requirement of solving a non-linear problem. This requires using numerical methods. In 1999, D. Lutterkort, J. Peter, and U. Reif proved in [6] that degree reduction of Bézier curves in the L_2 norm equals best Euclidean approximation of Bézier points. The generalization to the constrained case was done by Ahn et. al. in [1] while the author in [2] studied the discrete cases. The idea of using the Jacobi basis and Bernstein was introduced by Rababah et.al in 2007, see [12]. The existing methods to find degree reduction have many issues including: stability issues, accuracy, complexity, accumulate round-off errors, experiencing ill-conditioned systems, leading to a singularity, difficulty and indirect in applying the methods. A. Rababah and S. Mann presented also in [13] linear G^1, G^2, and G^3-multiple degree reduction methods for Bézier curves. We observe a significant progress in degree reduction of Bézier curves when Rababah and Ibrahim introduced the weighted G^0 and weighted G^1, weighted G^1 and weighted G^2 methods in [8, 9, 10] respectively. Simsek Y and Gunay M use the application of Bernstein polynomials to find an explicit polynomial representation of Bézier curves in [14]. Because of the new development in modern design and digital technology, the author in [3] makes use of Intrusion Identification with an idea of Bézier curve to verify the automated offline signature. Bézier curves are more user friendly and significant in the area of medical image practice, [5] use Bihistogram Bezier curve contrast enhancement to improve medical image visual appearance. Recently, the authors in [15] present Geometric Degree Reduction of Bézier Curves.

Most of degree reducing methods consider application of free parameters and conditions at the end points, but in this paper, we present the Chebyshev weight with the problem of degree reduction of Bézier curves. So that it gives more weight to the center of the curve. It is more suitable to consider degree reduction with the weight function $w[t] = 2t(1-t)$, $t \in [0,1]$. The obtained result yields better approximation at the center of the curves with minimum error. This method has not been studied and appeared in the literature yet; and this paper fills this vacancy.

2. PRELIMINARIES

Defination 1. *A Bézier curve $P_n(t)$ of degree n is defined algebraically as follows:*

$$P_n(t) = \sum_{i=0}^{n} p_i B_i^n(t) \quad 0 \le t \le 1, \tag{1}$$

where

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad i = 0, 1, \ldots, n,$$

are the Bernstein polynomials of degree n, and $p_0, p_1, \ldots, p_n$ are refers to as the Bézier control points of the Bézier curve.

The product of two Bernstein polynomials with the weight function $2t(1-t)$ is given by

$$B_i^m(t)B_j^n(t)2t(1-t) = \frac{2\binom{m}{i}\binom{n}{j}}{\binom{m+n+2}{i+j+1}} B_{i+j+1}^{m+n+2}(t).$$

We define the Gram matrix $G_{m,n}$ as the $(m+1)\times(n+1)$-matrix, whose elements are given by

$$g_{ij} = \int_0^1 B_i^m(t)B_j^n(t)2t(1-t)dt = \frac{2\binom{m}{i}\binom{n}{j}}{(m+n+3)\binom{m+n+2}{i+j+1}}, \quad i=0,\ldots,m, \quad j=0,1,\ldots,n. \tag{2}$$

The matrix $G_{m,m}$ is positive definite, symmetric, and real [13].

The process of analysing the continuity and studying the behaviour of two curves with some geometric properties is refers to as Geometric continuity. It does not depend on their parametrization and is denoted by G^k. Geometric continuity provides extra free parameters, see [7, 13] that are used to minimize the error.

Defination 2. *Bézier curves P_n and R_m are said to be G^k-continuous at $t=0,1$ if there exists a strictly increasing parametrization $s(t):[0,1]\to[0,1]$ with $s(0)=0, s(1)=1$, and*

$$R_m^{(i)}(t) = P_n^{(i)}(s(t)), \quad t=0,1, \quad i=0,1,\ldots,k. \tag{3}$$

3. DEGREE REDUCTION OF BÉZIER CURVES

Degree reduction can be defined as a method of approximating a given Bézier curve of degree n by a Bézier curve of degree $m, m<n$. Degree reduction is approximative process in nature and exact degree reduction is ordinarily not possible. In this paper, our aim is to find a Bézier curve $R_m(t)$ of degree m with control points $\{r_i\}_{i=0}^m$ that approximates a given Bézier curve $P_n(t)$ of degree n with control points $\{p_i\}_{i=0}^n$, where $m<n$. The Bézier curve R_m has to satisfy the following two conditions:

(1) P_n and R_m are G^3-continuous at the end points and
(2) the L_2-error between P_n and R_m is minimum.

We can write two Bézier curves $P_n(t)$ and $R_m(t)$ in matrix form as.

$$P_n(t) = \sum_{i=0}^n p_i B_i^n(t) =: B_n P_n, \quad 0\le t\le 1, \tag{4}$$

$$R_m(t) = \sum_{i=0}^m r_i B_i^m(t) =: B_m R_m, \quad 0\le t\le 1, \tag{5}$$

where B_n, B_m are the row matrices involving the Bernstein polynomials of degree n, m, respectively, and P_n and R_m are the column matrices involving the Bézier points of degrees n and m, respectively.

In this research, we consider the weighted L_2-norm to measure distance between the Bézier curves $P_n(t)$ and $R_m(t)$; therefore, the error term is given by

$$\begin{aligned} \varepsilon &= \int_0^1 ||B_nP_n - B_mR_m||^2 2t(1-t)dt \\ &= \int_0^1 ||B_nP_n - B_m^cR_m^c - B_m^fR_m^f||^2.2t(1-t)dt. \end{aligned} \tag{6}$$

4. METHODOLOGY

We form the linear systems and solve them for G^3-degree reduction. The control polygons of the Bézier curve are extended into their x and y subparts. Therefore, the vectors of our system of equations are r_k^x, $r_k^y, k=4,\ldots,m-4$, η_0,η_1,ζ_0 and ζ_1. The following vectors are defined to express the linear system in explicit form:

$$\begin{aligned} P^C &= [p_0^x,\ldots,p_n^x,p_0^y,\ldots,p_n^y]^t, \\ R^F &= [r_4^x,\ldots,r_{m-4}^x,r_4^y,\ldots,r_{m-4}^y,\eta_0,\eta_1,\zeta_0,\zeta_1]^t, \\ R^C &= [r_0^x,\ r_1^x,\ v_2^x,\ v_3^x,\ v_{m-3}^x,\ v_{m-2}^x,\ r_{m-1}^x,\ r_m^x, \\ &\quad\ r_0^y,\ r_1^y,\ v_2^y,\ v_3^y,\ v_{m-3}^y,\ v_{m-2}^y,\ r_{m-1}^y,\ r_m^y]^t. \end{aligned}$$

For G^3-degree reduction, we have to decompose each of r_2,r_3, and r_{m-3},r_{m-2} into a constant part and a part involving η_0,η_1,ζ_0 and ζ_1. We define the following notations as the constant parts of r_2,r_3, and r_{m-3},r_{m-2}:

$$v_2 = 2r_1 - r_0 + \frac{n(n-1)}{m(m-1)}\Delta^2 p_0, \tag{7}$$

$$v_{m-2} = 2r_{m-1} - r_m + \frac{n(n-1)}{m(m-1)}\Delta^2 p_{n-2}, \tag{8}$$

$$v_3 = 3v_2 - 3r_1 + r_0 + \frac{n(n-1)(n-2)}{m(m-1)(m-2)}\Delta^3 p_0, \tag{9}$$

$$v_{m-3} = 3v_{m-2} - 3r_{m-1} + r_m - \frac{n(n-1)(n-2)}{m(m-1)(m-2)}\Delta^3 p_{n-3}. \tag{10}$$

Let η_0^* and η_1^* be the coefficients of η_0 and η_1 in r_3 and r_{m-3} respectively:

$$\eta_0^* = 3\Delta p_0 + \frac{3(n-1)}{(m-2)}\Delta^2 p_0, \tag{11}$$

$$\eta_1^* = 3\Delta p_{n-1} - \frac{3(n-1)}{(m-2)}\Delta^2 p_{n-2}. \tag{12}$$

The following vectors are defined to express the linear system in explicit form:

$$P_n^C = [p_0^x, ..., p_n^x, p_0^y, ..., p_n^y]^t,$$
$$R_m^F = [r_2^x, \ldots, r_{m-2}^x, r_2^y, \ldots, r_{m-2}^y, \delta_0, \delta_1]^t,$$
$$R_m^C = [r_0^x,\ v_1^x,\ v_{m-1}^x,\ r_m^x,\ \ r_0^y,\ v_1^y,\ v_{m-1}^y,\ r_m^y]^t.$$

We can find our unknowns as, see [13]

$$R_m^F = (G_{m,m}^F)^{-1}\Big(G_{m,n}^{PC} P_n^C - G_{m,m}^C R_m^C\Big), \tag{13}$$

where

$$G_{m,n}^p := G_{m,n}(4, \ldots, m-4; 0, 1, \ldots, n),$$
$$G_{m,m}^c := G_{m,m}(4, \ldots, m-4; 0, 1, 2, 3, m-3, m-2, m-1, m),$$
$$G_{m,m}^f := G_{m,m}(4, \ldots, m-4; 4, \ldots, m-4),$$

and $G_{m,n}(\ldots;\ldots)$ is the sub-matrix of $G_{m,n}$ formed by the indicated rows and columns.

5. APPLICATIONS AND ILLUSTRATIONS

We apply the methods on an example to support the effectiveness of the new method. Considering the Bézier curve $P_n(t)$ of degree 19 with the following control points, see [13]:
$\mathbf{P}_0 = (37, 38)$, $\mathbf{P}_1 = (43, 37)$, $\mathbf{P}_2 = (39, 27)$, $\mathbf{P}_3 = (29, 26)$, $\mathbf{P}_4 = (23, 36)$,
$\mathbf{P}_5 = (26, 50)$, $\mathbf{P}_6 = (45, 56)$, $\mathbf{P}_7 = (58, 47)$, $\mathbf{P}_8 = (58, 29)$, $\mathbf{P}_9 = (46, 14)$,
$\mathbf{P}_{10} = (26, 6)$, $\mathbf{P}_{11} = (5, 15)$, $\mathbf{P}_{12} = (0, 40)$, $\mathbf{P}_{13} = (3, 58)$, $\mathbf{P}_{14} = (24, 68)$,
$\mathbf{P}_{15} = (50, 75)$, $\mathbf{P}_{16} = (79, 69)$, $\mathbf{P}_{17} = (79, 36)$, $\mathbf{P}_{18} = (65, 12)$, $\mathbf{P}_{19} = (50, 0)$.
This curve is reduced to Bézier curve $R_m(t)$ of degree 8.
Fig. 1 depicts the original curve in solid-black, weighted G^1-degree reduction in dashed-green curve, while at the right: the curve is reduced to degree 8 with C^1-method in (dashed-red), G^1-method in (dashed-blue) and real curve in (black).
Fig. 2 illustrates the real curve in black, weighted G^2-degree reduction in dashed-red curve, while at the right: the curve is reduced to degree 8 with C^2-method in (dashed-red), C^1/G^2-method in (dashed-green) and real curve in (black).
Fig. 3 illustrates the real curve in solid-black, weighted G^3-degree reduction in dashed-red curve, while at the right: the curves is reduced to degree 8 with C^3-method in (dashed-red), C^1/G^3-method in (dashed-green) and real curve in (black).
Regarding the error functions in **Fig. 4** long dashed-blue, dotted-green and dashed-red curves represent weighted G^1, G^2 and G^3-degree reduction respectively, while on the Right: it is reduced to degree 8 with G^1-method in (blue), C^1/G^2-method in (green) and weighted C^1/G^3-method in red and using (black) no constraints.
Fig. 5 illustrates the real curve in solid-black, weighted G^3-degree reduction in dashed-green curve, while at

the right: the curve is reduced to degree 9 with C^3-method in (dashed-red), C^1/G^3-method in (dot-green), and real curve in (black).

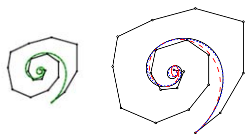

FIGURE 1. Left: The curve of degree 19 reduced to degree 8 with weighted G^1 in (dot-green) and real curve in (black). Right: It is reduced to degree 8 with C^1 method in (dashed-red), G^1-method in (dashed-blue) and real curve in (black).

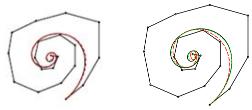

FIGURE 2. Left: The curve of degree 19 reduced to degree 8 with weighted G^2 in (dot-red) and real curve in (black). Right: It is reduced to degree 8 with C^2 method in (dashed-red), C^1/G^2 method in (dot-green) and real curve in (black).

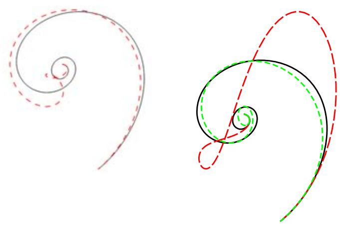

FIGURE 3. Left: The curve of degree 19 reduced to degree 8 with weighted G^3 in (dashed-red) and real curve in (black). Right: It is reduced to degree 8 with C^3 method in (dashed-red), C^1/G^3 method in (dashed-green) and real curve in (black).

Fig.	Errors of existing G^1, C^1/G^2, C^1/G^3	Errors of proposed Weighted G^1, G^2, G^3
1 and 4	Error of existing G^1 : 5.1	Error of proposed weighted G^1: 0.9
2 and 4	Error of existing C^1/G^2: 12.5	Error of proposed weighted G^2: 3.7
3 and 4	Error of existing C^1/G^3: 29.2	Error of proposed weighted G^3: 8

Table 1: Comparison with other existing methods

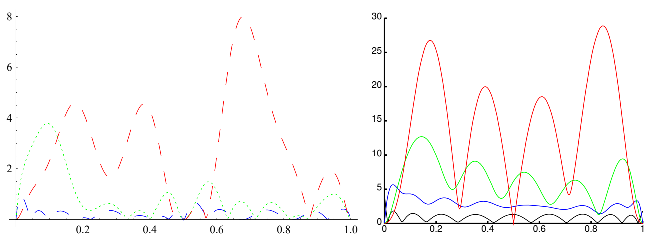

FIGURE 4. Error plots: Left: Degree 19 reduced to degree 8 with weighted G^3-method in (dashed-red), weighted G^2 method in (dotted-green) and weighted G^1 method in (long-dashed-blue). Right: It is reduced to degree 8 with G^1-method in (blue), C^1/G^2 method in (green) and C^1/G^3 method in (red) and using (black) no constraints.

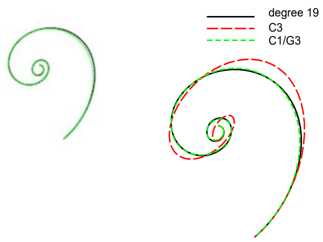

FIGURE 5. Left: The curve of degree 19 reduced to degree 9 with weighted G^3-method in (dashed-green) and real curve in (solid- black). Right: It is reduced to degree 9 with C^3-method in (dashed red), C^1/G^3 method in (dashed-green), and real curve in (black).

6. CONCLUSION

By considering weighted G^3- method and comparing it with the existing methods of degree reduction of Bézier curves. The Numerical results obtained from our examples show that our method outperforms and guarantee lesser error than the existing methods.

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