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Research article

Poly-Genocchi polynomials and its applications

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Abstract: In this paper, we discussed some new properties on the newly defined family of Genocchi polynomials, called poly-Genocchi polynomials. These polynomials are extensions from the Genocchi polynomials via generating function involving polylogarithm function. We succeeded in deriving the analytical expression and obtained higher order and higher index of poly-Genocchi polynomials for the first time. We also showed that the orthogonal version of poly-Genocchi polynomials could be presented as multiple shifted Legendre polynomials and Catalan numbers. Furthermore, we extended the determinant form and recurrence relation of shifted Genocchi polynomials sequence to shifted poly-Genocchi polynomials sequence. Then, we apply the poly-Genocchi polynomials to solve the fractional differential equation, including the delay fractional differential equation via the operational matrix method with a collocation scheme. The error bound is presented, while the numerical examples show that this proposed method is efficient in solving various problems.

Keywords: poly-Genocchi polynomials; shifted Genocchi polynomials sequence; Catalan numbers; determinant form and recurrence relation; fractional differential equation
Mathematics Subject Classification: 11B83, 26A33

1. Introduction

Genocchi polynomial is one of the important polynomials in the family of Appell polynomials. Moreover, many interesting patterns and number sequences arise in the field of combinatory [1–3]. Recently, Genocchi polynomials had been widely used to solve the various fractional calculus problems since the first research work in [4]. Following from there, the Genocchi polynomials or Genocchi wavelets were used in solving fractional differential equations with delay [5], fractional differential equations [6], system of Volterra integro-differential equation [7], fractional diffusion wave equation and fractional Klein–Gordon equation [8], nonlocal anti-periodic boundary value problem of arbitrary fractional order [9], fractional Abel differential equation [10],variable order fractional optimal control problems [11]. On the other hand, researchers in combinatory extended Genocchi polynomials' study to poly-Genocchi polynomials [12] and Apostol-Genocchi polynomials [13]. Many interesting properties had been discovered. For example, the bivariate poly-Genocchi polynomial is related to Stirling numbers of the second kind [14] and generalized Laguerre poly-Genocchi polynomials [15].

This paper will discuss some new properties of these poly-Genocchi polynomials, which can be extended from the properties of Genocchi polynomials. We successfully derived the analytical expression to obtain higher-order and higher index of poly-Genocchi polynomials using the lower order and lower index of poly-Genocchi polynomials. We also showed that the orthogonal version of the poly-Genocchi polynomials is multiple of the shifted Legendre polynomials. The multiple is related to the Catalan numbers. We extended the determinant form and recurrence relation of shifted Genocchi polynomials sequence recently introduced in [16] to shifted poly-Genocchi polynomials sequence. Apart from this, we use the poly-Genocchi polynomials to derive the operational matrix and apply it to solve fractional differential equations. This kind of operational matrix is widely used in solving fractional calculus problems such as fractional partial differential equation [17], linear Fredholm-Volterra integro-differential equations [18], nonlinear variable-order time fractional reaction–diffusion equation involving Mittag-Leffler kernel [19], fractional strongly nonlinear Duffing oscillators [20]. More specifically, we apply this new poly-Genocchi polynomial operational matrix to solve some benchmark problems and compare the results as in [21] for delay fractional differential equation (or so called generalized fractional pantograph equations).

The rest of the paper is organized as follows. Section 2 briefly explains the poly-Genocchi polynomials and their new properties while discussing the orthogonal version of poly-Genocchi polynomials via the Gram-Schmidt process in subsection 2.1. Apart from that, we explain the procedure to obtain the shifted poly-Genocchi polynomials sequence. Then, in Section 3, we apply the poly-Genocchi polynomials to derive the operational matrix of the derivative. The error bound of the method is also presented in this section. Moreover, the numerical examples are presented in Section 4. Last, we provide a summary and some recommendations in Section 5.

2. Properties of poly-Genocchi polynomials

According to [22], the poly-Genocchi polynomials, $G_n^{(k)}(x)$ can be obtained using the generating function in (2.1).

$$\frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}, \tag{2.1}$$

where $Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$ denotes the k^{th} polylogarithm function. For $x = 0$, we obtain the poly-Genocchi numbers, $g_n^{(k)} = G_n^{(k)}(0)$ of index k, where k is a positive integer. By using Eq (2.1), we obtain the following first few poly-Genocchi polynomials of index k, $G_n^{(k)}(x)$.

$$
\begin{aligned}
&G_1^{(k)}(x) = 1,\\
&G_2^{(k)}(x) = 2x - 2 + \frac{2}{2^k},\\
&G_3^{(k)}(x) = 3x^2 + \left(\frac{6}{2^k} - 6\right)x + \frac{6}{3^k} - \frac{9}{2^k} + \frac{5}{2},\\
&G_4^{(k)}(x) = 4x^3 + \left(\frac{12}{2^k} - 12\right)x^2 + \left(\frac{24}{3^k} - \frac{36}{2^k} + 10\right)x + \frac{24}{4^k} - \frac{48}{3^k} + \frac{26}{2^k} - 2.
\end{aligned}
\tag{2.2}
$$

For $k = 1$, Eq (2.2) reduces to Genocchi polynomials given below.

$$
\begin{aligned}
&G_1(x) = 1,\\
&G_2(x) = 2x - 1,\\
&G_3(x) = 3x^2 - 3x,\\
&G_4(x) = 4x^3 - 6x^2 + 1.
\end{aligned}
\tag{2.3}
$$

Some of the properties of Genocchi polynomials can be extended to poly-Genocchi polynomials, such as

$$
\begin{aligned}
G_n^{(k)}(x) &= \sum_{r=0}^{n} \binom{n}{r} g_{n-r}^{(k)} x^r\\
&= g_n^{(k)} + \sum_{r=0}^{n-1} \binom{n}{r} g_r^{(k)} x^{n-r},\\
\frac{dG_n^{(k)}(x)}{dx} &= nG_{n-1}^{(k)}(x), \quad n \geq 1,\\
\frac{d^2G_n^{(k)}(x)}{dx^2} &= n(n+1)G_{n-2}^{(k)}(x), \quad n \geq 2,\\
\int_0^x G_n^{(k)}(x)dx &= \frac{G_{n+1}^{(k)}(x) - g_{n+1}^{(k)}}{n+1}.
\end{aligned}
\tag{2.4}
$$

Here, we refer the readers to other interesting properties of poly-Genocchi polynomials in [22, 23], the properties of Genocchi polynomials in [24–26] and other families of Genocchi related polynomials in [27, 28].

For function $f(x)$ in an arbitrary element of $L^2[0, 1]$, ones can approximate the function in terms of poly-Genocchi polynomials as in (2.5):

$$
f(x) \approx f^*(x) = \sum_{n=1}^{N} c_n G_n^{(k)}(x) = \mathbf{C}\mathbf{G}^{(k)}(x)^T,
\tag{2.5}
$$

where $\mathbf{C} = [c_1, c_2, \cdots, c_N]$, $\mathbf{G}^{(k)}(x) = [G_1^{(k)}(x),\ G_2^{(k)}(x), \cdots, G_N^{(k)}(x)]$ and $\mathbf{C}$ or c_n can be calculated using the following equation,

$$
\mathbf{C} = [c_1, c_2, \cdots, c_N] = \frac{\left\langle f(x), \mathbf{G}^{(k)}(x)\right\rangle}{\left\langle \mathbf{G}^{(k)}(x), \mathbf{G}^{(k)}(x)\right\rangle},
\tag{2.6}
$$

where $\left\langle \mathbf{G}^{(k)}(x), \mathbf{G}^{(k)}(x)\right\rangle$ is an $N \times N$ matrix.

The poly-Genocchi polynomials, $G_{n+1}^{(k)}(x)$ can be obtained if the lower degree n and lower index k are known. We now introduce the following theorem.

Theorem 1. *The poly-Genocchi polynomials, $G_n^{(k)}(x)$ can be determined as follows:*

$$\begin{aligned} G_{n+1}^{(k)}(x) = &-\frac{1}{2}G_n^{(k-1)}(x) + \sum_{i=0}^{\lfloor n/2 \rfloor+1} \frac{G_{2i}^{(k-1)}(x)}{(2i)!}\frac{n!B_{n-2i+1}}{(n-2i+1)!} \\ &- \frac{1}{2(n+1)} \sum_{\substack{i\geq 0,\\ i+j=n+1}}^{n} (-1)^{n+i}\binom{n+1}{i} G_i^{(k)}(x)G_j + xG_n^{(k)}(x), \;\; \textit{for} \;\; n = \textit{odd}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} G_{n+1}^{(k)}(x) = &-\frac{1}{2}G_n^{(k-1)}(x) + \sum_{i=0}^{n/2} \frac{G_{2i+1}^{(k-1)}(x)}{(2i+1)!}\frac{n!B_{n-2i}}{(n-2i)!} \\ &- \frac{1}{2(n+1)} \sum_{\substack{i\geq 0,\\ i+j=n+1}}^{n} (-1)^{n+i}\binom{n+1}{i} G_i^{(k)}(x)G_j + xG_n^{(k)}(x), \;\; \textit{for} \;\; n = \textit{even}, \end{aligned} \tag{2.8}$$

where B_n is Bernoulli number and G_n is Genocchi number obtained using $G_n = 2(1-2^n)B_n$.

Proof. Suppose the generating function of poly-Genocchi polynomials as follows:

$$\frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}. \tag{2.9}$$

Both sides of Eq (2.9) is then differentiated w.r.t t, which yields

$$\frac{2Li_{k-1}(1-e^{-t})e^{xt}}{e^t(1-e^{-t})(e^t+1)} - \frac{2Li_k(1-e^{-t})e^{xt}e^t}{(e^t+1)^2} + \frac{2Li_k(1-e^{-t})e^{xt}x}{(e^t+1)} = \sum_{n=0}^{\infty} G_{n+1}^{(k)}(x)\frac{t^n}{n!}. \tag{2.10}$$

For the LHS of (2.10) and using (2.9), we denote B_n as n^{th} Bernoulli number, while G_n as the Genocchi number, which yields

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} G_n^{(k-1)}(x)\frac{t^n}{n!}\right)\left(\frac{1}{e^t-1}\right) - \left(\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}\right)\left(\frac{e^t}{e^t+1}\right) + \left(\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}\right)x \\ &= \left(\sum_{n=0}^{\infty} G_n^{(k-1)}(x)\frac{t^n}{n!}\right)\left(-\frac{1}{2} + \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}t^{2n-1}\right) \\ &- \left(\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{2}\frac{G_{n+1}}{n+1}\frac{t^n}{n!}\right) + \left(\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}\right)x. \end{aligned} \tag{2.11}$$

The first two terms in (2.11) are expanded, where after some algebraic manipulation, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\sum_{n=0}^{\infty} G_n^{(k-1)}(x)\frac{t^n}{n!} + \sum_{\substack{n=0,\\ n=odd}}^{\infty}\sum_{i=0}^{\lfloor n/2\rfloor+1} \frac{G_{2i}^{(k-1)}(x)}{(2i)!}\frac{n!B_{n-2i+1}}{(n-2i+1)!}\frac{t^n}{n!} \\
&+\sum_{\substack{n=0,\\ n=even}}^{\infty}\sum_{i=0}^{n/2}\frac{G_{2i+1}^{(k-1)}(x)}{(2i+1)!}\frac{n!B_{n-2i}}{(n-2i)!}\frac{t^n}{n!} \\
&-\sum_{n=0}^{\infty}\frac{1}{2(n+1)}\sum_{\substack{i\geq 0,\\ i+j=n+1}}^{n}(-1)^{n+i}\binom{n+1}{i}G_i^{(k)}(x)G_j\frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}\right)x.
\end{aligned}
\tag{2.12}
$$

By equating coefficients, when n is odd, we obtain

$$
\begin{aligned}
G_{n+1}^{(k)}(x) = &-\frac{1}{2}G_n^{(k-1)}(x) + \sum_{i=0}^{\lfloor n/2\rfloor+1}\frac{G_{2i}^{(k-1)}(x)}{(2i)!}\frac{n!B_{n-2i+1}}{(n-2i+1)!} \\
&-\frac{1}{2(n+1)}\sum_{\substack{i\geq 0,\\ i+j=n+1}}^{n}(-1)^{n+i}\binom{n+1}{i}G_i^{(k)}(x)G_j + xG_n^{(k)}(x).
\end{aligned}
\tag{2.13}
$$

On the other hand, for even n, we have

$$
\begin{aligned}
G_{n+1}^{(k)}(x) = &-\frac{1}{2}G_n^{(k-1)}(x) + \sum_{i=0}^{n/2}\frac{G_{2i+1}^{(k-1)}(x)}{(2i+1)!}\frac{n!B_{n-2i}}{(n-2i)!} \\
&-\frac{1}{2(n+1)}\sum_{\substack{i\geq 0,\\ i+j=n+1}}^{n}(-1)^{n+i}\binom{n+1}{i}G_i^{(k)}(x)G_j + xG_n^{(k)}(x).
\end{aligned}
\tag{2.14}
$$

This completes the proof. □

2.1. Orthogonal version of poly-Genocchi polynomials

Here, we briefly explain the Gram-Schmidt process for the poly-Genocchi polynomials. Note that we have $G_0^{(k)}(x) = 0$ and also $G_1^{(k)}(x) = 1$. Suppose also that $\phi_1(x), \cdots, \phi_q(x)$ are orthogonal version of poly-Genocchi polynomials obtained from Gram-Schmidt process in which the polynomial is orthogonal with respect to the inner product $\langle f, g\rangle = \int_0^1 w(x)f(x)g(x)dx$. Then

$$
\phi_{q+1}(x) = G_{q+1}^{(k)}(x) - \sum_{i=1}^{q}\lambda_i\phi_i(x)
\tag{2.15}
$$

satisfies $\left\langle \phi_{q+1}, \phi_j\right\rangle = \int_0^1 w(x)\phi_{q+1}(x)\phi_j(x)dx = 0, j = 0, 1, \cdots, q$ with $\lambda_j = \frac{\left\langle G_{q+1}^{(k)},\phi_j\right\rangle}{\left\langle \phi_j,\phi_j\right\rangle}$.

Obviously, we have $\phi_0(x) = 0, \phi_1(x) = 1$ and shifted Legendre polynomials, $P_0(x) = 1$. Here, we compare the poly-Genocchi polynomials of degree $q + 1$ with the shifted Legendre polynomials of degree q since both of them have same highest power of x. By using Gram-Schmidt process as in (2.15) with $q = 1$, we obtain

$$\begin{aligned}
\phi_2(x) &= G_2^{(k)}(x) - \sum_{i=1}^{1} \lambda_i \phi_i(x) \\
&= \sum_{r=0}^{2} \binom{2}{r} g_{2-r}^{(k)} x^r - \frac{\int_0^1 \sum_{r=0}^{2} \binom{2}{r} g_{2-r}^{(k)} x^r \phi_1(x) dx}{\int_0^1 (\phi_1(x))^2 dx} \phi_1(x) \\
&= \sum_{r=0}^{2} \binom{2}{r} g_{2-r}^{(k)} \left(x^r - \frac{1}{r+1} \right) \\
&= 2x - 1.
\end{aligned}$$

This is the same as degree 1 shifted Legendre polynomials, $P_1(x) = 2x - 1$. In other words, the orthogonal version of poly-Genocchi polynomials, $\phi_2(x)$, regardless of the k value in (2.1) or (2.2), is the multiple of 1 for degree 1 shifted Legendre polynomials after Gram-Schmidt process. Now, by using Gram-Schmidt process as in (2.15) with $q = 2$, we obtain

$$\begin{aligned}
\phi_3(x) &= G_3^{(k)}(x) - \sum_{i=1}^{2} \lambda_i \phi_i(x) \\
&= \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} x^r - \frac{\int_0^1 \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} x^r \phi_2(x) dx}{\int_0^1 (\phi_2(x))^2 dx} \phi_2(x) - \frac{\int_0^1 \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} x^r \phi_1(x) dx}{\int_0^1 (\phi_1(x))^2 dx} \phi_1(x) \\
&= \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} x^r - 3 \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} \left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+1}}{r+1} \right]_0^1 (2x-1) - \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} \left[\frac{x^{r+1}}{r+1} \right]_0^1 \\
&= \sum_{r=0}^{3} \binom{3}{r} g_{3-r}^{(k)} \left(x^r - \frac{3r(2x-1)}{(r+2)(r+1)} - \frac{1}{r+1} \right) \\
&= 3x^2 - 3x + \frac{1}{2}.
\end{aligned}$$

After this Gram-Schmidt process, we obtain the orthogonal version of poly-Genocchi polynomials, $\phi_3(x)$ in multiple of 2 for degree 2 shifted Legendre polynomials, $P_2(x)$. Upon continuing the Gram-Schmidt process as in (2.15) with $q = 3$ yields

$$
\begin{aligned}
\phi_4(x) &= G_4^{(k)}(x) - \sum_{i=1}^{3} \lambda_i \phi_i(x) \\
&= \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} x^r - \frac{\int_0^1 \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} x^r \phi_3(x) dx}{\int_0^1 (\phi_3(x))^2 dx} \phi_3(x) \\
&- \frac{\int_0^1 \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} x^r \phi_2(x) dx}{\int_0^1 (\phi_2(x))^2 dx} \phi_2(x) - \frac{\int_0^1 \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} x^r \phi_1(x) dx}{\int_0^1 (\phi_1(x))^2 dx} \phi_1(x) \\
&= \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} x^r - 20 \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} \left[\frac{3x^{r+3}}{r+3} - \frac{3x^{r+2}}{r+2} + \frac{1}{2} \frac{x^{r+1}}{r+1} \right]_0^1 \left(3x^2 - 3x + \frac{1}{2} \right) \\
&- 3 \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} \left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+1}}{r+1} \right]_0^1 (2x-1) - \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} \left[\frac{x^{r+1}}{r+1} \right]_0^1 \\
&= \sum_{r=0}^{4} \binom{4}{r} g_{4-r}^{(k)} \left(x^r - \frac{5r(r-1)(6x^2-6x+1)}{(r+3)(r+2)(r+1)} - \frac{3r(2x-1)}{(r+2)(r+1)} - \frac{1}{r+1} \right) \\
&= 4x^3 - 6x^2 + \frac{12}{5} x - \frac{1}{5}.
\end{aligned}
$$

After this Gram-Schmidt process, we obtain the poly-Genocchi polynomials, $\phi_4(x)$ in multiple of 5 for degree 3 shifted Legendre polynomials, $P_3(x)$. By using similar algebra manipulation, for $\phi_5(x) = G_5^{(k)}(x) - \sum_{i=1}^{4} \lambda_i \phi_i(x)$, we obtain $\phi_5(x) = 5x^4 - 10x^3 + \frac{45}{7}x^2 - \frac{10}{7}x + \frac{1}{14}$ which is the multiple of 14 for degree 4 shifted Legendre polynomials, $P_4(x)$. We summarize the results as shown in Table 1. These multiples are indeed the Catalan numbers $1, 1, 2, 5, 14...$ which given by $C_n = \frac{1}{n+1}\binom{2n}{n}$. More generally, we have

$$
\begin{aligned}
\phi_{q+1}(x) &= G_{q+1}^{(k)}(x) - \sum_{i=1}^{q} \lambda_i \phi_i(x) \\
&= \sum_{r=0}^{q} \binom{q}{r} g_{q-r}^{(k)} \left(x^r - \sum_{m=0}^{q-1} \frac{(2m+1)(\prod_{a=0}^{m-1}(r-a))(P_m(x))}{\prod_{a=0}^{m}(r+a+1)} \right) \\
&= \frac{1}{q+1}\binom{2q}{q} P_q(x).
\end{aligned}
$$

In conclusion, for the orthogonal version of poly-Genocchi polynomials, $\phi_q(x)$ obtained via Gram-Schmidt process, we have

$$
\phi_{q+1}(x) = \frac{1}{q+1}\binom{2q}{q} P_q(x), \tag{2.16}
$$

where $\frac{1}{q+1}\binom{2q}{q}$ are the Catalan numbers, $C(q)$.

Table 1. Existing of Catalan numbers in the Gram-Schmidt process for poly-Genocchi polynomials, $\phi_q(x)$.

Orthogonal poly-Genocchi polynomials, $\phi_{q+1}(x)$	Shifted Legendre polynomials, $P_q(x)$	Multipler
$\phi_1(x) = 1$	$P_0(x) = 1$	1
$\phi_2(x) = 2x - 1$	$P_1(x) = 2x - 1$	1
$\phi_3(x) = 3x^2 - 3x + \frac{1}{2}$	$P_2(x) = 6x^2 - 6x + 1$	2
$\phi_4(x) = 4x^3 - 6x^2 + \frac{12}{5}x - \frac{1}{5}$	$P_3(x) = 20x^3 - 30x^2 + 12x - 1$	5
$\phi_5(x) = 5x^4 - 10x^3 + \frac{45}{7}x^2 - \frac{10}{7}x + \frac{1}{14}$	$P_4(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1$	14

2.2. *Shifted poly-Genocchi polynomials sequence*

This subsection extends the determinant form and recurrence relation of shifted Genocchi polynomials sequence recently introduced in [16] to shifted poly-Genocchi polynomials sequence. Similar to [16], we shift the order of poly-Genocchi polynomials from n to $n + 1$, i.e. we have $Gs_n^{(k)}(x) = G_{n+1}^{(k)}(x)$, where $Gs_n^{(k)}(x)$ denotes shifted poly-Genocchi polynomials sequence. We now have the following lemma.

Lemma 1. *For $n > 0$, the determinant form and recurrence relation of shifted poly-Genocchi polynomials sequence, $Gs_n^{(k)}(x)$ is given by*

$$Gs_n^{(k)}(x) = \frac{(-1)^n}{\prod_{i=0}^{n} s_{i,i}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ s_{0,0} & s_{1,0} & s_{2,0} & \cdots & s_{n-1,0} & s_{n,0} \\ 0 & s_{1,1} & s_{2,0} & \cdots & s_{n-1,1} & s_{n,1} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & s_{n-1,n-1} & s_{n,n-1} \end{vmatrix} \tag{2.17}$$

and

$$Gs_n^{(k)}(x) = \frac{1}{s_{n,n}} \left(x^n - \sum_{j=0}^{n-1} s_{n,j} Gs_j^{(k)}(x) \right). \tag{2.18}$$

The procedure to obtain the values for $s_{i,j}$ follows from Francesco A. Costabile et al. [16], summarized as follows:

Step 1: From $G_n^{(k)}(x) = \sum_{r=0}^{n} \binom{n}{r} g_{n-r}^{(k)} x^r$, the poly-Genocchi number, $g_i^{(k)}$ is obtained. Hence, we calculate the lower triangular Toeplitz matrix, T_G with entries

$$t_{i,j} = \frac{g_{i+1-j}^{(k)}}{(i + 1 - j)!}.$$

Step 2: The upper triangular, S, can be obtained via

$$S = D_2^{-1}\, T_G^{-1}\, D_1^{-1},$$

where $D_1 = \operatorname{diag}\{(i+1)! \mid i = 0, 1, \ldots\}$, while $D_2 = \operatorname{diag}\{1/i! \mid i = 0, 1, \ldots\}$.

For example, the determinant form of shifted poly-Genocchi polynomials sequence, $Gs_3^{(2)}(x)$ ($k = 2, n = 3$) and $Gs_4^{(3)}(x)$ ($k = 3, n = 4$) are given by

$$Gs_3^{(2)}(x) = \frac{(-1)^3}{\frac{1}{4!}} \begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & \frac{3}{4} & \frac{59}{72} & \frac{95}{96} \\ 0 & \frac{1}{2} & \frac{3}{4} & \frac{59}{48} \\ 0 & 0 & \frac{1}{3} & \frac{3}{4} \end{vmatrix}, \quad Gs_4^{(3)}(x) = \frac{(-1)^4}{\frac{1}{5!}} \begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & \frac{7}{8} & \frac{863}{864} & \frac{2909}{2304} & \frac{25345241}{15552000} \\ 0 & \frac{1}{2} & \frac{7}{8} & \frac{863}{576} & \frac{2909}{1152} \\ 0 & 0 & \frac{1}{3} & \frac{7}{8} & \frac{863}{432} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{7}{8} \end{vmatrix}.$$

It is easy to see that the above determinant forms give the poly-Genocchi polynomials as follows:

$$\begin{aligned} G_4^{(2)}(x) &= Gs_3^{(2)}(x) = 4x^3 - 9x^2 + \frac{11}{3}x + \frac{2}{3}, \\ G_5^{(3)}(x) &= Gs_4^{(3)}(x) = 5x^4 - \frac{35}{2}x^3 + \frac{575}{36}x^2 - \frac{55}{72}x - \frac{4819}{3600}. \end{aligned} \tag{2.19}$$

3. Operational matrix based on poly-Genocchi polynomials

This section derives the new operational matrix based on poly-Genocchi polynomials and applies it to solve the fractional differential equations. This new operational matrix is called the generalization of the Genocchi operational matrix developed in [29].

Theorem 2. *Suppose $\boldsymbol{G}^{(k)}(x)$ is the poly-Genocchi vector $\boldsymbol{G}^{(k)}(x) = [G_1^{(k)}(x),\ G_2^{(k)}(x), \cdots, G_N^{(k)}(x)]^T$ and let $\alpha > 0$. Then,*

$$D^\alpha \boldsymbol{G}^{(k)}(x)^T = \boldsymbol{P}^\alpha \boldsymbol{G}^{(k)}(x)^T, \tag{3.1}$$

where P^α is an $N \times N$ operational matrix of fractional derivative of order α in Caputo sense and is defined as follows:

$$\boldsymbol{P}^{(\alpha)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum\limits_{r=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,r,1} & \sum\limits_{r=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,r,2} & \cdots & \sum\limits_{r=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \theta_{\lceil\alpha\rceil,r,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{r=\lceil\alpha\rceil}^{n} \theta_{n,r,1} & \sum\limits_{r=\lceil\alpha\rceil}^{n} \theta_{n,r,2} & \cdots & \sum\limits_{r=\lceil\alpha\rceil}^{n} \theta_{n,r,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{r=\lceil\alpha\rceil}^{N} \theta_{N,r,1} & \sum\limits_{r=\lceil\alpha\rceil}^{N} \theta_{N,r,2} & \cdots & \sum\limits_{r=\lceil\alpha\rceil}^{N} \theta_{N,r,N} \end{bmatrix},$$

where $\theta_{n,r,j}$ is given by

$$\theta_{n,r,j} = \frac{n! g_{n-r}^{(k)}}{(n-r)!\Gamma(r+1-\alpha)} c_j. \tag{3.2}$$

Here $g_n^{(k)}$ is the poly-Genocchi number and c_j can be obtained from the inner product via (2.6).

Proof. From (2.4), we can write the poly-Genocchi polynomials in analytical form, where its fractional derivative is given as in (3.3).

$$D^{\alpha} G_n^{(k)}(x) = \sum_{r=0}^{n} \frac{n! g_{n-r}^{(k)}}{(n-r)! r!} D^{\alpha} x^r \;=\; \sum_{r=\lceil\alpha\rceil}^{n} \frac{n! g_{n-r}^{(k)}}{(n-r)! \Gamma(r+1-\alpha)} x^{r-\alpha}. \tag{3.3}$$

Let $f(x) = x^{r-\alpha}$. By using truncated poly-Genocchi polynomials, we obtain $f(x) = \sum_{j=1}^{N} c_j G_j^{(k)}(x)$. Substituting this into (3.3) yields

$$\begin{aligned} D^{\alpha} G_n^{(k)}(x) &= \sum_{j=1}^{N} \left(\sum_{r=\lceil\alpha\rceil}^{n} \frac{n! g_{n-r}^{(k)}}{(n-r)! \Gamma(r+1-\alpha)} c_j \right) G_j^{(k)}(x) \\ &= \sum_{j=1}^{N} \left(\sum_{r=\lceil\alpha\rceil}^{n} \theta_{n,r,l} \right) G_j^{(k)}(x), \end{aligned} \tag{3.4}$$

where $\theta_{n,r,l}$ is given in (3.2). Rewriting (3.4) in the vector form, we have

$$D^{\alpha} G_n^{(k)}(x) = \left[\sum_{r=\lceil\alpha\rceil}^{n} \theta_{\lceil\alpha\rceil,r,1} \quad \sum_{r=\lceil\alpha\rceil}^{n} \theta_{\lceil\alpha\rceil,r,2} \quad \cdots \quad \sum_{r=\lceil\alpha\rceil}^{n} \theta_{\lceil\alpha\rceil,r,N} \right] \mathbf{G}^{(k)}(x), \tag{3.5}$$

where $n = \lceil\alpha\rceil \cdots N$. For $n = 1, \cdots, \lceil\alpha\rceil - 1$, we have

$$D^{\alpha} G_n^{(k)}(x) = [0, 0, \cdots 0]\, \mathbf{G}^{(k)}(x), \quad n = 1, \cdots, \lceil\alpha\rceil - 1. \tag{3.6}$$

Hence, by combining (3.5) and (3.6), the poly-Genocchi operational matrix as in (3.1) is proven. □

3.1. Error bound

In this subsection, we briefly explain the error bound for the function approximation for arbitrary $f(x)$ by using poly-Genocchi polynomials.

Theorem 3. *For the arbitrary function $f(x) \in C^{n+1}[0,1]$ and*

$$Y = Span\{G_1^{(k)}(x), G_2^{(k)}(x), \cdots, G_N^{(k)}(x)\}.$$

Let $\boldsymbol{C}^T \boldsymbol{G}^{(k)}(x)$ be the best approximation of $f(x)$ out of Y, we then have

$$\|f(x) - \boldsymbol{C}^T \boldsymbol{G}^{(k)}(x)\| \le \frac{h^{\frac{2n+3}{2}} M}{(n+1)! \sqrt{2n+3}}, \quad x \in [x_i, x_{i+1}] \subseteq [0,1], \tag{3.7}$$

where $M = \max_{x \in [x_i, x_{i+1}]} |f^{(n+1)}(x)|$ and $h = x_{i+1} - x_i$.

Proof. By using Taylor's series, we can write

$$y_1(x) = f(x_i) + f'(x_i)(x - x_i) + f''(x_i)\frac{(x - x_i)^2}{2!} + \cdots + f^{(n)}(x_i)\frac{(x - x_i)^n}{n!}.$$

If we truncate the Taylor's series, the following bound may be obtained,

$$|f(x) - y_1(x)| \leq |f^{(n+1)}(\xi_x)|\frac{(x - x_i)^{n+1}}{(n+1)!},$$

where $\xi_x \in [x_i, x_{i+1}]$.

Since $\mathbf{C}^T\mathbf{G}^{(k)}(x)$ is the best approximation of $f(x)$ out of Y and $y_1(t) \in Y$, then from $\forall y(x) \in Y,\ \|f(x) - f^*(x)\|_2 \leq \|f(x) - y(x)\|_2$, we have

$$\begin{aligned} \|f(x) - \mathbf{C}^T\mathbf{G}^{(k)}(x)\|_2^2 &\leq \|f(x) - y_1(x)\|_2^2 \\ &= \int_{x_i}^{x_{i+1}} |f(s) - y_1(s)|^2 ds \\ &\leq \int_{x_i}^{x_{i+1}} \|f^{(n+1)}(\xi_x)\|^2 \left(\frac{(s - x_i)^{n+1}}{(n+1)!}\right)^2 ds \\ &\leq \frac{h^{2n+3}M^2}{((n+1)!)^2(2n+3)}. \end{aligned} \tag{3.8}$$

Taking the square root of both sides of (3.8) yields

$$\|f(x) - \mathbf{C}^T\mathbf{G}^{(k)}(x)\| \leq \frac{h^{\frac{2n+3}{2}}M}{(n+1)!\sqrt{2n+3}}.$$

This proofs the error bound inequality as in (3.7). □

In short, for each sub interval $[x_i, x_{i+1}]$, $i = 1, 2, \cdots, n$, $f(x)$ has a local error bound of $O(h^{\frac{2n+3}{2}})$ while for the whole interval, $[0, 1]$, $f(x)$ has a global error of $O(h^{\frac{2n+1}{2}})$.

3.2. Collocation scheme based on poly-Genocchi operational matrix

In this subsection, we use the collocation scheme based on the poly-Genocchi operational matrix to numerically solve the fractional differential equation. This kind of approach replaces symbol by symbol, i.e. replacing fractional derivative, D^α with the operational matrix, $\mathbf{P}^\alpha$. This approach is also the same if we intend to solve the integer order differential equations, i.e. when $\alpha = 1$. To do this, we have the following procedure:

Step 1: We first approximate $y(x)$ using poly-Genocchi polynomials as follows:

$$y(x) = \sum_{r=1}^{N} c_r G_r^{(k)}(x) = \mathbf{C}\mathbf{G}^{(k)}(x)^T, \tag{3.9}$$

where $\mathbf{C} = [c_1, \quad c_2, \cdots, c_N]$ is an unknown vector that need to be determined. If we want to approximate fractional derivative for $y(x)$, we replace it by poly-Genocchi operational matrix as in (3.1) yielding

$$D^{\alpha} y(x) \simeq \mathbf{C}\mathbf{P}^{(\alpha)}\mathbf{G}^{(k)}(x)^T. \tag{3.10}$$

For the initial and boundary conditions, we can replace $y(0) = a$ with $\mathbf{C}\mathbf{G}^{(k)}(0)^T - a = 0$ and $y(1) = b$ with $\mathbf{C}\mathbf{G}^{(k)}(1)^T - b = 0$.

Step 2: Substituting (3.9) and (3.10) into the fractional differential equation, it collocates at the collocation points $x_i = \frac{i}{N}$, $i = 1, 2, \cdots, N-2$. Together with initial and boundary condition, we have N algebraic equations. We solve this system of algebraic equations with Newton's iterative method to obtain the value for $\mathbf{C} = [c_1, \quad c_2, \cdots, c_N]$. Thus, the solution of fractional differential equation is obtained using (3.9).

4. Application in solving fractional differential equation

In this section, we solve some fractional differential equations to illustrate the applicability and accuracy of these poly-Genocchi polynomials. We achieve this by using the collocation scheme and the fractional derivative by employing an operational matrix based on poly-Genocchi polynomials. This operational matrix is the generalization of different indexes of Genocchi polynomials. All the numerical computations are carried out using Maple.

Example 1. *Consider a simple fractional differential equation, given by*

$$D^{(1/2)} y(x) + y'(x) = \frac{8}{3}\frac{x^{3/2}}{\sqrt{\pi}} + 2x, \tag{4.1}$$

with initial condition $y(0) = 0$. The exact solution is given by $y(x) = x^2$.

This problem is solved using collocation scheme with $N = 4, 8$ and poly-Genocchi polynomials for $k = 2$ and $k = 5$. The absolute errors for the Example 1 are shown in Table 2. From the table, although the numerical scheme is simple and easy to use, the solution is accurate. Using different k values (i.e. $k = 2, 5$) of poly-Genocchi polynomials will give the same numerical result.

Example 2. *Consider the following fractional delay differential equation as in [21, 30].*

$$D^{(\frac{5}{2})} y(x) = -y(x) - y(x - 0.5) + h(x), \quad x \in [0, 1], \tag{4.2}$$

with initial condition $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $h(x) = \frac{\Gamma(4)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} + x^3 + (x - 0.5)^3$.

The exact solution is $y(x) = x^3$. Here, we compare our results with those in [30] using the generalized Laguerre-Gauss collocation scheme with Laguerre parameters β and in [21] using a collocation scheme based on Genocchi operational matrix. Using $N = 4$ with poly-Genocchi polynomials $k = 2$ and $k = 5$, respectively, and following the procedure as in Example 2 [21], we obtained the result as in Table 3. Obviously, the proposed method with poly-Genocchi operational matrix gives better results.

Table 2. Absolute errors for proposed method for Example 1.

	Absolute errors, $N = 4$		Absolute errors, $N = 8$	
x	$k = 2$	$k = 5$	$k = 2$	$k = 5$
0	1.00000E-15	6.00000E-16	1.00000E-15	2.99000E-15
0.1	4.77187E-04	4.77187E-04	4.23183E-06	4.23183E-06
0.2	6.79231E-04	6.79231E-04	1.07341E-05	1.07341E-05
0.3	6.81968E-04	6.81968E-04	2.92267E-06	2.92267E-06
0.4	5.61237E-04	5.61237E-04	7.80106E-07	7.80106E-07
0.5	3.92875E-04	3.92875E-04	6.45866E-06	6.45866E-06
0.6	2.52718E-04	2.52718E-04	7.37434E-06	7.37434E-06
0.7	2.16605E-04	2.16605E-04	6.47144E-07	6.47144E-07
0.8	3.60373E-04	3.60373E-04	2.74148E-06	2.74148E-06
0.9	7.59859E-04	7.59859E-04	8.16082E-06	8.16082E-06
1.0	1.49090E-03	1.49090E-03	1.38949E-04	1.38949E-04

Table 3. Comparison of the absolute errors obtained by the proposed method with those in [21, 30] for Example 2.

	[30]	[21]	Proposed Method	
x	$\beta = 5, N = 22$	$N = 4$	$N = 4, k = 2$	$N = 4, k = 5$
0	-	-	5.00000E-15	4.00000E-15
0.1	6.273E-06	6.17040E-09	5.97970E-09	5.97969E-09
0.2	3.892E-05	4.93630E-08	4.78376E-08	4.78376E-08
0.3	1.023E-04	1.66600E-07	1.61452E-07	1.61452E-07
0.4	1.901E-04	3.94910E-07	3.82701E-07	3.82701E-07
0.5	2.944E-04	7.71300E-07	7.47462E-07	7.47462E-07
0.6	4.088E-04	1.33280E-06	1.29161E-06	1.29161E-06
0.7	5.306E-04	2.11640E-06	2.05104E-06	2.05104E-06
0.8	6.597E-04	3.15920E-06	3.06160E-06	3.06160E-06
0.9	7.977E-04	4.49820E-06	4.35920E-06	4.35920E-06
1.0	9.468E-04	6.17040E-06	5.97970E-06	5.97970E-06

Example 3. *We consider the Lane-Emden equation up to the fractional order. It has been widely used in describing the thermal distribution profile in the human head [31] and radial stress on a rotationally symmetric shallow membrane cap [32]. The equation is given by:*

$$D^{(\alpha)}y(x) + \frac{1}{x}y'(x) + e^{y(x)} = 0, \quad x \in [0, 1] \tag{4.3}$$

with initial and boundary conditions $y'(0) = 0, \ \ y(1) = 0.$

The exact solution for $\alpha = 2$ is given by $y(x) = 2\ln\left(\frac{B+1}{Bx^2+1}\right)$, where $B = 3 - 2\sqrt{2}$. By using $N = 8$ with poly-Genocchi polynomials when $k = 2$, we obtain the approximate solution for $\alpha = 2, 1.9, 1.8$ as in Figure 1.

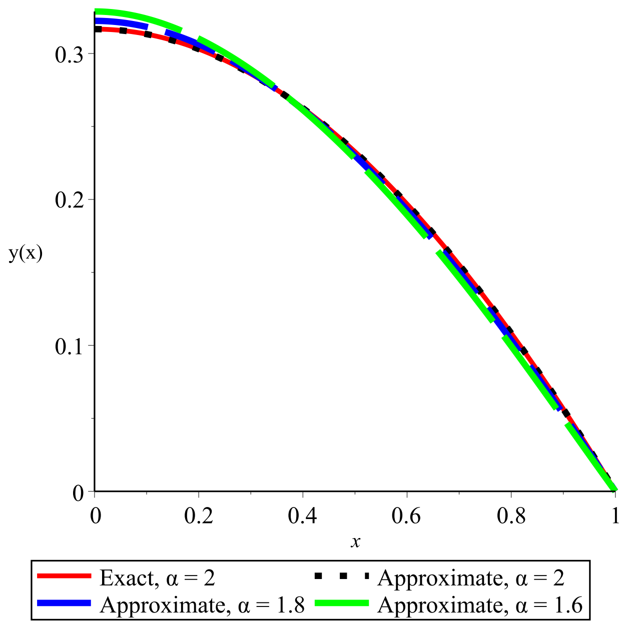

Figure 1. Comparison of approximate solution and exact solution for Example 3.

The numerical results are compared with the solution obtained in [33] using Modified Adomian Decomposition Method (MADM). It is obtained by combining between the Adomian Decomposition Method and collocation approach based on quintic B-spline basis function. The result in Table 4 shows that our proposed method with fewer terms is comparable with the result in [33].

Table 4. Comparison of the Maximum Absolute Errors (MAE) obtained by MADM, Quintic B-spline and proposed method for Example 3.

Interval $[0, 0.1]$ MADM [33]	Interval $[0, 0.1]$ Proposed method	Interval $[0.1, 1]$ Quintic B-spline [33]	Interval $[0.1, 1]$ Proposed method
$n = 10$, 6.56E-10	$N = 12$, 6.96E-10	$N = 18$, 1.47E-09	$N = 12$, 6.80E-10
$n = 12$, 1.55E-11	$N = 15$, 1.84E-13	$N = 36$, 9.45E-11	$N = 15$, 2.16E-13

Example 4. *We consider the Bratu type equation as in [34]. Our proposed method are not only able to solve the integer order derivative for Bratu type equation, but also can solve the fractional Bratu type equation efficiently. This Bratu type equation are widely used in the fuel ignition model [35], the Chandrashekhar model [36]. Meanwhile, fractional Bratu type equations are studied for the problem arising in electro-spun organic nanofibers elaboration [37].*

Here, we extend the Bratu type equation discuss in [34] to fractional order derivative as follow:

$$D^{(\alpha)}y(x) + \lambda e^{y(x)} = 0, \quad x \in [0, 1] \tag{4.4}$$

with initial and boundary conditions $y(0) = 0, \ \ y(1) = 0.$

The exact solution for $\alpha = 2$ is given by $y(x) = -2\ln\left(\frac{cosh((x-0.5)\frac{\theta}{2})}{cosh(\frac{\theta}{4})}\right)$, where θ is the solution of

$\theta - \sqrt{2\lambda}\cosh(\frac{\theta}{4}) = 0$. Using $N = 8$ with poly-Genocchi polynomial when $k = 2$ and let $\lambda = 1$ in the Eq (4.4), we obtain the approximate solution for $\alpha = 2, 1.9, 1.8$ as in Figure 2.

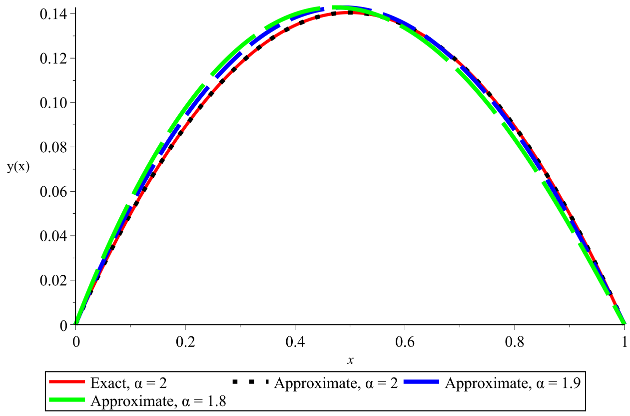

Figure 2. Comparison of approximation solution and exact solution for Example 4.

When $\alpha = 2, \lambda = 2$, by using $N = 8$, we compare our approximate solution with the iteration method developed in [34]. The result is presented as in Table 5. Our solution is comparable with published result in [34].

Table 5. Comparison of the proposed method with iteration method [34] for Example 4 when $\lambda = 2$.

x	Absolute error Iteration method, $n = 6$ [34]	Absolute error Proposed method, $N = 8$
0	0	1.91463E-10
0.2	6.9297E-05	3.73495E-05
0.4	1.0775E-04	9.31864E-05
0.6	1.0775E-04	1.38795E-04
0.8	6.9297E-05	1.70014E-04
1.0	7.9936E-17	1.91463E-10

5. Conclusions

In this work, we investigated the new properties of poly-Genocchi polynomials, $G_n^{(k)}(x)$, with any positive integer, k. When $k = 1$, it reduces to Genocchi polynomials. We successful derived the analytical expression to obtain a higher order and higher index of poly-Genocchi polynomials. We show that the orthogonal version of poly-Genocchi polynomials is the multiple of shifted Legendre polynomials. Interestingly, the multiple is given by Catalan numbers. We also extended the determinant form and recurrence relation of shifted Genocchi polynomials sequence introduced in [16] to shifted poly-Genocchi polynomials sequence. We introduced the poly-Genocchi operational matrix for the first time, where the error bound for this new method is presented. Using a collocation scheme, we are able to solve the fractional differential equation and fractional delay differential equation. The numerical examples have shown that this proposed method is highly efficient and easy to use. Using few terms of poly-Genocchi polynomials in our proposed method can give more accurate results than existing methods. The method can be easily extended to solve more complicated problems such as those in [38, 39].

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Conflict of interest

Authors declare that there is no conflict of interests regarding the publication of the paper.

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