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Shifted Genocchi Polynomials Operational Matrix for Solving Fractional Order Stiff System

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Abstract. In this paper, we solve the fractional order stiff system using shifted Genocchi polynomials operational matrix. Different than the well known Genocchi polynomials, we shift the interval from $[0, 1]$ to $[1, 2]$ and name it as shifted Genocchi polynomials. Using the nice properties of shifted Genocchi polynomials which inherit from classical Genocchi polynomials, the shifted Genocchi polynomials operational matrix of fractional derivative will be derived. Collocation scheme are used together with the operational matrix to solve some fractional order stiff system. From the numerical examples, it is obvious that only few terms of shifted Genocchi polynomials is sufficient to obtain result in high accuracy.

Keywords: Fractional order stiff system, operational matrix, shifted Genocchi polynomials.

1. Introduction

A system is said to be stiff when it's solution entails components that change at significantly different rates to give changes in the independent variable. The stiff system can be represented by a differential equation. The areas of chemical engineering, biochemistry, life sciences and nonlinear mechanics are some of the sources of stiff problems, which normally governed by some ordinary differential equation. In this research we are going to consider fractional order stiff systems of the form

$$D^\alpha y_j(x) = f(x, y_1(x), y_2(x), \dots, y_n(x)) = g_j(x), \quad y_j(0) = \beta_j \quad (1)$$

where β_j are constants and $j = 1, 2, \dots, n$.

In numerical analysis for solving stiff systems or stiff system in fractional derivative order, the choice of the numerical solution step size is rather important. In this situation, large step size will lose some fast changing properties of the system, meanwhile small step size may causing round off errors. This will make the solution obtained using some numerical methods may not stable as small change in independent variables will case drastic change in the dependent variable. Furthermore, if we have the nonlinear problem, this stiffness problem will cause much more complicated for ones to find the appropriate solution. On the other hand, many stiff problem do not have analytical solution, or the analytical is not easy to found.



Hence, reliable numerical methods are always needed. Hence, in this paper, we intend to propose a new numerical scheme for solving this stiff problem in fractional order derivative. The fractional derivative is defined using the definition of Caputo.

In recent years, researchers increased interest and developed several methods to solve stiff system up to fractional order. Among that, including using Haar wavelets to solve linear and nonlinear stiff systems of ordinary differential equations [1, 2]. Apart from that, Decomposition method (ADM) was applied to obtain analytical solution on some stiff systems [3]. In [4], Aminikhah combined Laplace transform and perturbation method to solve the systems of ordinary differential equations with having stiff characteristic. On the other hand, Alshbool et al. [5] used multistage Bernstein polynomials to obtain the solution of the arbitrary order stiff system. Besides that, some numerical as well as analytical methods have been developed to tackle this stiff systems, which include in [6, 7]. Different than existing methods, in this paper, we will extend from an important member of Appell polynomials family called Genocchi polynomial to shifted Genocchi polynomial and use it to find the approximation solution of the fractional order stiff system through collocation method. With using collocation scheme and via some numerical examples, we able to show the applicability and high accuracy of our proposed method.

In other aspect, operational matrix method had been used to solve some fractional calculus problem, it is because the method is easy to use especially with the help of mathematical software and yet able to give the solution in high accuracy. Among that including solving fractional integro-differential equation in Fredholm type with Bernoulli operational matrix [8] and solving fractional optimal control problem by using Genocchi operational matrix [9], solving fractional order Brusselator by using Legendre wavelet operational matrix [10], solving fractional integro-differential equation by using Jacobi wavelet operational matrix of fractional integration [11]. In this paper, different than previous published results of the operational matrix method which focus on interval $[0, 1]$, here, we shifted the interval to $[1, 2]$. With successful in this shifting, we hope in future can shift the interval to the wider interval.

2. Preliminaries

This section, will briefly explain the fractional derivative defined by Caputo.

Definition 2.1 *The derivative of fractional order due to Caputo, D^α is defined as:*

$$D^\alpha f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{d^m f}{ds^m} (x - s)^{\alpha - m + 1} ds, \quad m - 1 < \alpha \leq m \text{ and } m \in \mathbb{N}. \quad (2)$$

For Caputo fractional derivatives, these properties are rather important.

$$D^\alpha C = 0, \quad \text{where } C \text{ is constant.} \quad (3)$$

$$D^\alpha x^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - \alpha)} x^{\lambda - \alpha}, \quad \lambda > \alpha, \quad (4)$$

where $\lceil \alpha \rceil$ is the ceiling function and $\lfloor \alpha \rfloor$ is the floor function.

3. Shifted Genocchi Polynomials and Some Properties

The Genocchi polynomials, $G_n(x)$ of order n , are defined in interval $[0, 1]$, through the exponential generating functions

$$\frac{2re^{xr}}{e^r + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{r^n}{n!}, \quad (|r| < \pi). \tag{5}$$

Expanding the $\sum_{n=0}^{\infty} G_n(x) \frac{r^n}{n!}$ in the equation (5) in the series form, we obtain

$$G_n(x) = \sum_{k=0}^n \binom{n}{k} G_k x^{n-k}. \tag{6}$$

Normally, this Genocchi polynomial is defined in the interval from 0 to 1. Hence, if the value of x is shifted from the interval $[0, 1]$ to the interval $[1, 2]$, then we define the shifted Genocchi polynomial $S(t)$ by changing $x = t - 1$. Thus, $G_n(t - 1) = S_n(t)$, hence the first few terms are:

$$\begin{aligned} S_1(t) &= 1, \\ S_2(t) &= 2t - 3, \\ S_3(t) &= 3t^2 - 9t + 6, \\ S_4(t) &= 4t^3 - 18t^2 + 24t - 9, \\ S_5(t) &= 5t^4 - 30t^3 + 60t^2 - 45t + 10, \\ S_6(t) &= 6t^5 - 45t^4 + 120t^3 - 135t^2 + 60t - 9, \\ S_7(t) &= 7t^6 - 63t^5 + 210t^4 - 315t^3 + 210t^2 - 63t + 14. \end{aligned}$$

Definition 3.1 The shifted Genocchi polynomials $S_n(t)$ of order n is defined over the interval $[1, 2]$ as:

$$\begin{aligned} S_n(t) &= G_n\left(\frac{t-1}{1}\right) = \sum_{k=0}^n \binom{n}{k} g_{n-k} \left(\frac{t-1}{1}\right)^k \\ &= \sum_{r=0}^n \binom{n}{r} s_{n-r} t^r, \end{aligned} \tag{7}$$

where $s_{n-r} = \sum_{k=r}^n \frac{(-1)^{k-r} g_{n-k} \binom{n}{k} \binom{k}{r}}{1^k \binom{n}{r}}$ is the shifted Genocchi number and let $S_0(t) = 0$.

One can easily check that for the shifted Genocchi polynomials have the following properties which is inherit from the Genocchi polynomials.

$$\frac{dS_n(t)}{dt} = nS_{n-1}(t), \quad n \geq 1, \tag{8}$$

$$S_n(1) + S_n(2) = 0, \quad n > 1. \tag{9}$$

The analytical form of the shifted Genocchi polynomials is given by

$$S_i(t) = \sum_{k=0}^i \sum_{r=0}^k \frac{(-1)^{k-r} i! G_{i-k}}{(i-k)!(k-r)! r!} t^r. \tag{10}$$

3.1. Function Approximation using Shifted Genocchi Polynomials

Suppose that $H = L^2[1, 2]$ and we have following conditions,

(i) $\{S_1(t), S_2(t), \dots, S_N(t)\} \subset H$ is the set containing different order of shifted Genocchi polynomials.

(ii) $Y = Span\{S_1(t), S_2(t), \dots, S_N(t)\}$ is a vector space which is clearly finite dimensional.

Let f be any element of H , then f has a typical excellent approximation in Y , let it be f^* , such that $\forall y \in Y \quad \|f - f^*\|_2 \leq \|f - y\|_2$. Since $f^* \in Y$, then there exist the unique coefficients c_1, c_2, \dots, c_N such that

$$f \approx f^* = \sum_{n=1}^N c_n S_n(t) = \mathbf{CS}(t),$$

where $\mathbf{C} = [c_1, c_2, \dots, c_N]$ and $\mathbf{S}(t)$ is the shifted Genocchi vector given by $[S_1(t), S_2(t), \dots, S_N(t)]$.

For finding the coefficients c_n , the following lemma plays the role.

Lemma 3.2 Assume that $f \in L^2[1, 2]$, is any function approximated by the series $\sum_{n=1}^N c_n S_n(t)$, then the coefficients c_n for $n = 1, 2, \dots, N$ can be calculated using the equation as follows:

$$c_n = \frac{1}{2n!} (f^{(n-1)}(1) + f^{(n-1)}(2)).$$

Here, the power $(n - 1)$ represents the $(n - 1)^{th}$ derivative of function f in $f^{(n-1)}(x)$.

Proof. Suppose $f(x) \approx \sum_{n=1}^N c_n S_n(t)$, then, we have

$$\begin{aligned} f(1) &= c_1 S_1(1) + c_2 S_2(1) + c_3 S_3(1) + \dots + c_N S_N(1), \\ f(2) &= c_1 S_1(2) + c_2 S_2(2) + c_3 S_3(2) + \dots + c_N S_N(2). \end{aligned}$$

With applying equation (8) and (9), one has $f(1) + f(2) = c_1(S_1(1) + S_1(2)) + c_2(S_2(1) + S_2(2)) + c_3(S_3(1) + S_3(2)) + \dots + c_N(S_N(1) + S_N(2)) = 2c_1$. Note that $S_2(1) + S_2(2) = 2$ and using equation (9), (i.e. $S_n(1) + S_n(2) = 0, \quad n > 1$), thus, $c_1 = \frac{1}{2}(f(1) + f(2))$.

For obtaining $f^{(1)}(1), f^{(1)}(2)$, we first get the series as $f(t) = c_1 S_1(t) + c_2 S_2(t) + \dots + c_N S_N(t)$, then differentiate it and later substitute when $t = 1$ or $t = 2$. Hence, we obtain $f^{(1)}(1) + f^{(1)}(2) = 2c_2(S_1(1) + S_1(2)) + 3c_3(S_2(1) + S_2(2)) + \dots + Nc_N(S_{N-1}(1) + S_{N-1}(2)) = 2c_2(2)$, thus, $c_2 = \frac{1}{2(2!)}(f^{(1)}(1) + f^{(1)}(2))$.

Using the similar process, for $f^{(2)}(1) + f^{(2)}(2) = 3(2)c_3(S_1(1) + S_1(2)) + \dots + N(N - 1)c_N(S_{N-2}(1) + S_{N-2}(2)) = 3(2)c_3(2)$, thus, $c_3 = \frac{1}{2(3!)}(f^{(2)}(1) + f^{(2)}(2))$, and for $f^{(3)}(1) + f^{(3)}(2) = 4(3)(2)c_4(S_1(1) + S_1(2)) + \dots + N(N - 1)(N - 2)c_N(S_{N-3}(1) + S_{N-3}(2)) = 4(3)(2)c_4(2)$, thus, $c_4 = \frac{1}{2(4!)}(f^{(3)}(1) + f^{(3)}(2))$. This procedure is continued until i times for $i = 1, 2, \dots, N$, then we will obtain

$$c_i = \frac{1}{2i!} (f^{(i-1)}(1) + f^{(i-1)}(2)).$$

The proof is completed.

This shifting from interval $[0, 1]$ to $[1, 2]$ is important as the formula for finding the c_n for classical Genocchi polynomials, which is $c_n = \frac{1}{2n!}(f^{(n-1)}(0) + f^{(n-1)}(1))$ may fail when the $f^{(n-1)}(0)$ is not defined.

Lemma 3.3 Let $S_i(t)$ be the shifted polynomial due to Genocchi then, $D^\alpha S_i(t) = 0$, $i = 1, \dots, [\alpha] - 1$, $\alpha > 0$.

Proof. From equation (10), we have $S_i(t) = \sum_{k=0}^i \sum_{r=0}^k \frac{(-1)^{k-r} i! G_{i-k}}{(i-k)!(k-r)!r!} t^r$.

Now, for $i = 1, \dots, [\alpha] - 1$ if $S_i(t) = C$, (C is any constant), then by using equation (3), we obtain $D^\alpha S_i(t) = 0$.

Otherwise, by using equation (4), one can easily see that $D^\alpha S_i(t) = 0$ for $i = 1, \dots, [\alpha] - 1$, this complete the proof of the Lemma.

4. Shifted Genocchi Operational Matrix of Fractional Derivative

We consider the shifted Genocchi vector, $\mathbf{S}(t)$, which can be written as $\mathbf{S}(t) = [S_1(t), S_2(t), \dots, S_N(t)]$, then using equation (8), the derivative of the shifted Genocchi vector $\mathbf{S}(t)$ can be written in the matrix form by $\frac{d\mathbf{S}(t)^T}{dt} = M\mathbf{S}(t)^T$ where

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & N & 0 \end{pmatrix}$$

M is $N \times N$ matrix and in this case, M is said to be the operational matrix of integer derivative.

If we continue to find the k^{th} derivative of $\mathbf{S}(t)$, we will be able to obtain

$$\frac{d^k \mathbf{S}(t)^T}{dt^k} = \mathbf{S}(t)(M^T)^k.$$

Now, the operational matrix of fractional order derivative using shifted Genocchi polynomials is shown in the following theorem.

Theorem 4.1 Suppose $\mathbf{S}(t)$ is a shifted Genocchi vector and let $\alpha > 0$. Then,

$$D^\alpha \mathbf{S}(t)^T = M^\alpha \mathbf{S}(t)^T \quad (11)$$

where D^α is $N \times N$ operational matrix of fractional derivative of order α which defined using Caputo derivative and can be written as follows:

$$M^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,k,1} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,k,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^i \rho_{i,k,1} & \sum_{k=\lceil\alpha\rceil}^i \rho_{i,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^i \rho_{i,k,N} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^N \rho_{N,k,1} & \sum_{k=\lceil\alpha\rceil}^N \rho_{N,k,2} & \cdots & \sum_{k=\lceil\alpha\rceil}^N \rho_{N,k,N} \end{pmatrix},$$

where $\rho_{i,k,l}$ can be calculated from

$$\rho_{i,k,l} = \sum_{r=0}^k \frac{(-1)^{k-r} i! G_{i-k} (f^{(l-1)}(1) + f^{(l-1)}(2))}{2l!(i-k)!(k-r)!\Gamma(r+1-\alpha)} \tag{12}$$

and G_{i-k} denotes the Genocchi number.

Proof. Using equation (4) and (10), we have:

$$D^\alpha S_i(t) = \sum_{k=0}^i \sum_{r=0}^k \frac{(-1)^{k-r} i! G_{i-k}}{(i-k)!(k-r)!r!} D^\alpha t^r = \sum_{k=\lceil\alpha\rceil}^i \sum_{r=0}^k \frac{(-1)^{k-r} i! G_{i-k}}{(i-k)!(k-r)!\Gamma(r+1-\alpha)} t^{r-\alpha}. \tag{13}$$

Let $f(t) = t^{r-\alpha}$, then the $f(t)$ can be approximated by truncated shifted Genocchi series. In this case, we have $f(t) = \sum_{l=0}^N a_l S_l(t)$. From the Lemma 3.2, we have $c_l = \frac{1}{2l!} (f^{(l-1)}(1) + f^{(l-1)}(2))$. Therefore, $f(t) = \sum_{l=0}^N \frac{1}{2l!} (f^{(l-1)}(1) + f^{(l-1)}(2)) S_l(t)$, putting this in (13), we have

$$D^\alpha S_i(t) = \sum_{l=0}^N \left(\sum_{k=\lceil\alpha\rceil}^i \sum_{r=0}^k \frac{(-1)^{k-r} i! G_{i-k} (f^{(l-1)}(1) + f^{(l-1)}(2))}{2l!(i-k)!(k-r)!\Gamma(r+1-\alpha)} \right) S_l(t) = \sum_{l=0}^N \left(\sum_{k=\lceil\alpha\rceil}^i \rho_{i,k,l} \right) S_l(t), \tag{14}$$

where $\rho_{i,k,l}$ is defined in (12). Putting (14) in form of vector, we get

$$D^\alpha S_i(t) = \left[\sum_{k=\lceil\alpha\rceil}^i \rho_{\lceil\alpha\rceil,k,1} \quad \sum_{k=\lceil\alpha\rceil}^i \rho_{\lceil\alpha\rceil,k,2} \quad \cdots \quad \sum_{k=\lceil\alpha\rceil}^i \rho_{\lceil\alpha\rceil,k,N} \right] S(t) \quad i = \lceil\alpha\rceil \cdots N. \tag{15}$$

Again by Lemma 3.3, we have

$$D^\alpha S_i(t) = [0, 0, \dots, 0] S(t) \quad i = 1, \dots, \lceil\alpha\rceil - 1. \tag{16}$$

Thus, putting together equation (15) and ((16), we obtain the result as in this theorem.

5. Procedure of Collocation Method for Solving Fractional Stiff System via Shifted Genocchi operational Matrix

In this section, we will briefly explain how we use the shifted Genocchi operational matrix of fractional derivatives to solve the stiff system as in (1) numerically. We achieve this by applying collocation method. To do this, we follows the following steps :

Step 1: Approximate $D^\alpha y_j(t)$, $y_j(t)$ and $g_j(t)$ $j = 1, 2, \dots, n$, by means of shifted Genocchi polynomials as follows:

$$y_j(t) = \sum_{k=0}^N c_{j,k} S_k(t) = \mathbf{C}_j \mathbf{S}(t)^T, \tag{17}$$

$$g_j(t) = \sum_{k=0}^N g_{j,k} S_k(t) = \mathbf{G}_j \mathbf{S}(t)^T, \tag{18}$$

where the vector $\mathbf{G}_j = [g_{j,1}, g_{j,2}, \dots, g_{j,N}]$ is known vector, but $\mathbf{C}_j = [c_{j,1}, c_{j,2}, \dots, c_{j,N}]$ is unknown vector.

Step 2: Now employing equation (11) on (17), we have

$$D^\alpha y_j(t) \simeq \mathbf{C}_j M^{(\alpha)} \mathbf{S}(t)^T, \quad j = 1, 2, \dots, n. \tag{19}$$

Step 3: Putting equations (17),(18) and (19) in (1), we get

$$\mathbf{C}_j M^{(\alpha)} \mathbf{S}(t)^T = f_j \left(t, \mathbf{C}_1 \mathbf{S}(t)^T, \mathbf{C}_2 \mathbf{S}(t)^T, \dots, \mathbf{C}_n \mathbf{S}(t)^T \right) = \mathbf{G}_j \mathbf{S}(t)^T \quad j = 1, 2, \dots, n. \tag{20}$$

The initial conditions gives

$$\mathbf{C}_j \mathbf{S}(t)^T = \beta_j \quad j = 1, 2, \dots, n. \tag{21}$$

Step 4: To obtain the solution of equation (1), we apply the collocation points $t_i = \frac{i}{N-1} + 1$, $i = 1, 2, \dots, N - 1$ on (20) to get

$$\mathbf{C}_j M^{(\alpha)} \mathbf{S}(t_i)^T = f_j \left(t_i, \mathbf{C}_1 \mathbf{S}(t_i)^T, \mathbf{C}_2 \mathbf{S}(t_i)^T, \dots, \mathbf{C}_n \mathbf{S}(t_i)^T \right) = \mathbf{G}_j \mathbf{S}(t_i)^T. \tag{22}$$

Step 5: Thus, equation (22) are $n(N - 1)$ algebraic equations. These equations together with equation (21) make $n(N)$ algebraic equations . This system of equations can be solved using any computer algebra system such as Maple, Mathematica directly. Consequently $y_j(t)$ given in equation (17) can be calculated.

6. Numerical Examples

Some numerical examples are given in this section to illustrate that our proposed method is not only easy to use, but also able to give good applicability and high accuracy of numerical result using this proposed scheme. We will present both linear and nonlinear stiff system which the derivative is not only limited to order 1, but also in fractional order. For this fractional stiff problem, when the derivative is not equal to order 1, mostly, there are no exact solution. Hence, for the usual practice, we will find the numerical solution when the fractional derivative is approaching 1, then the solution should be also approaching the exact solution when derivative is order 1.

Example 6.1 First lets consider the stiff system of fractional differential equation as follows:

$$\begin{cases} D^\alpha y_1(t) = -y_1(t) + 15y_2(t) + 15e^{-t}, \\ D^\alpha y_2(t) = 15y_1(t) - y_2(t) - 15e^{-t}, \end{cases} \tag{23}$$

subject to, $y_1(0) = y_2(0) = 1$.

Knowing that the exact solution of the stiff system is $y_1(t) = e^{-t}$ and $y_2(t) = e^{-t}$ when the derivative is $\alpha = 1$. Using the scheme described in previous sections with $N = 8$, we present the numerical solution and absolute error in Table 1. From the numerical result shown in the table, it is obvious that using this new operational matrix, able to give the result in high accuracy.

Table 1: Numerical results for Example 6.1

t	Exact solution $y_1(t) = y_2(t) = e^{-t}$	Proposed method $y_1(t)$	Proposed method $y_2(t)$	Absolute Error $y_1(t)$	Absolute Error $y_2(t)$
1.0	0.3678794412	0.3678794393	0.3678794409	1.90000E-09	3.00000E-10
1.1	0.3328710837	0.3328710838	0.3328710813	1.00000E-10	2.40000E-09
1.2	0.3011942119	0.3011942139	0.3011942110	2.00000E-09	9.00000E-10
1.3	0.2725317930	0.2725317942	0.2725317946	1.20000E-09	1.60000E-09
1.4	0.2465969639	0.2465969628	0.2465969655	1.10000E-09	1.60000E-09
1.5	0.2231301601	0.2231301580	0.2231301588	2.10000E-09	1.30000E-09
1.6	0.2018965180	0.2018965190	0.2018965152	1.00000E-09	2.80000E-09
1.7	0.1826835241	0.1826835290	0.1826835267	4.90000E-09	2.60000E-09
1.8	0.1652988882	0.1652988692	0.1652988868	1.90000E-08	1.40000E-09
1.9	0.1495686192	0.1495684550	0.1495685164	1.64200E-07	1.02800E-07
2.0	0.1353352832	0.1353346181	0.1353347294	6.65100E-07	5.53800E-07

Table 2: Numerical solution $y_1(t)$, exact solution $y_1(t) = e^{-2t}$ and absolute errors obtained by proposed method for Example 6.2

t	Exact solution	$y_1(t)$	Absolute Error $y_1(t)$
1.0	0.1353352832	0.1353530057	1.77225E-05
1.1	0.1108031584	0.1108176661	1.45077E-05
1.2	0.0907179533	0.0907298269	1.18736E-05
1.3	0.0742735782	0.0742833034	9.72515E-06
1.4	0.0608100626	0.0608180280	7.96532E-06
1.5	0.0497870684	0.0497935696	6.50118E-06
1.6	0.0407622040	0.0407675605	5.35656E-06
1.7	0.0333732700	0.0333776961	4.42615E-06
1.8	0.0273237224	0.0273259709	2.24848E-06
1.9	0.0223707719	0.0223638177	6.95417E-06
2.0	0.0183156389	0.0182768135	3.88254E-05

We also solve this example with $\alpha = 0.9, 0.95$ and the results are analogized in Figure 1 and 2, which clearly portrays that as α approaches 1, our numerical results approach the exact solution when $\alpha = 1$. This is acceptable since the solution of fractional order near to derivative order 1 is very close to the exact solution when $\alpha = 1$.

Example 6.2 Consider the non linear stiff system of FDE.

$$\begin{cases} D^\alpha y_1(t) = -1002y_1(t) + 1000y_2^2(t), \\ D^\alpha y_2(t) = y_1(t) - y_2(t) - y_2^2(t), \end{cases} \tag{24}$$

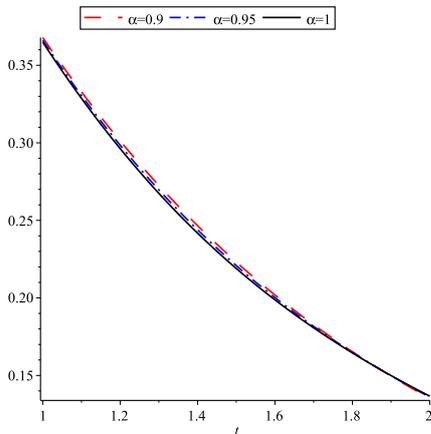


Figure 1: Numerical solutions for $y_1(t)$ when $\alpha = 0.9, 0.95$ and 1 for Example 6.1

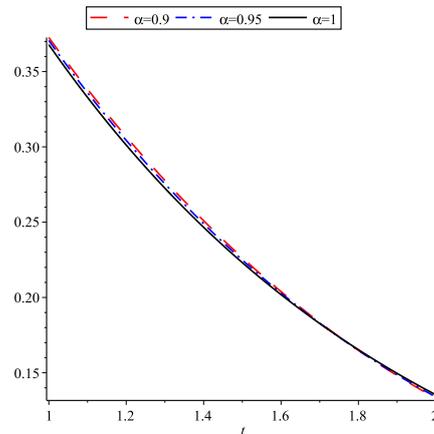


Figure 2: Numerical solutions for $y_2(t)$ when $\alpha = 0.9, 0.95$ and 1 for Example 6.1

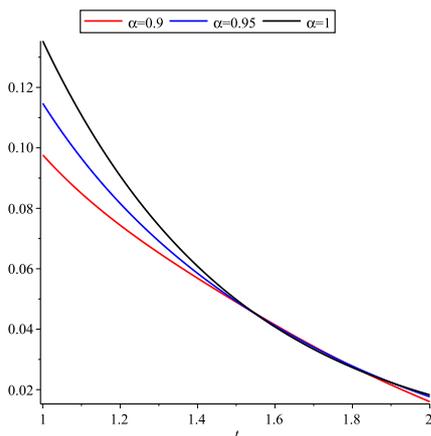


Figure 3: Numerical solutions for $y_1(t)$ when $\alpha = 0.9, 0.95$ and 1 for Example 6.2

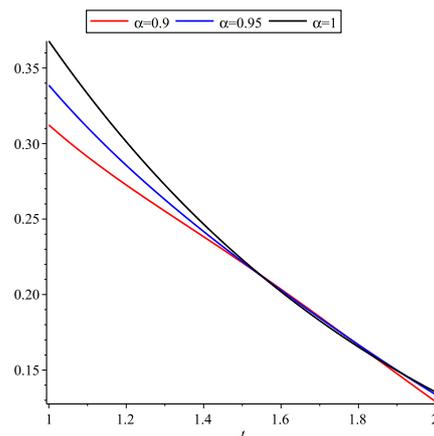


Figure 4: Numerical solutions for $y_2(t)$ when $\alpha = 0.9, 0.95$ and 1 for Example 6.2

subject to, $y_1(0) = y_2(0) = 1$.

Knowing that the exact solution of this system is $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$ when $\alpha = 1$. We solve this equation by the scheme described in previous section with $N = 8$. The numerical results and absolute errors are shown in Table 2 and Table 3, which shows that our method is of high accuracy in solving this kind of fractional stiff problem.

Table 3: Numerical solution $y_2(t)$, exact solution $y_2(t) = e^{-t}$ and absolute errors obtained by present method for example 6.2

t	Exact solution	$y_2(t)$	Absolute Error $y_2(t)$
1.0	0.3678794412	0.3679035207	2.40795E-05
1.1	0.3328710837	0.3328928717	2.17880E-05
1.2	0.3011942119	0.3012139264	1.97145E-05
1.3	0.2725317930	0.2725496316	1.78386E-05
1.4	0.2465969639	0.2466131050	1.61411E-05
1.5	0.2231301601	0.2231447652	1.46051E-05
1.6	0.2018965180	0.2019097330	1.32150E-05
1.7	0.1826835241	0.1826954818	1.19577E-05
1.8	0.1652988882	0.1653097073	1.08191E-05
1.9	0.1495686192	0.1495783928	9.77360E-06
2.0	0.1353352832	0.1353440420	8.75880E-06

We also solve this example when $\alpha = 0.9, 0.95$ and the comparison of the results with that when $\alpha = 1$ are shown in Figure 3 and 4 in which the figures confirm that when α approaches 1, our numerical results approach the exact solution when $\alpha = 1$. Again, this is acceptable since the solution of fractional order near to derivative order 1 is very close to the exact solution when $\alpha = 1$.

7. Conclusion

In this paper, we introduced the new family of Genocchi polynomials, which is called shifted Genocchi polynomials, where the interval are shifted from $[0, 1]$ to $[1, 2]$. Some new properties of this new shifted Genocchi polynomials were discussed. We derived the operational matrix of fractional derivative of this new shifted Genocchi polynomials. By using collocation method, we solved some fractional stiff problems. With only few terms of shifted Genocchi polynomials, we manage to obtain good result. For future work, we hope can apply this shifted Genocchi polynomials to more general interval, say for example $[a, b]$, where may having larger domain and at the same time, the nice properties of the Genocchi polynomials still hold. We hope can solve other kind of fractional calculus problems such as fractional partial differential equation [20], fractional partial integro-differential equations [21] and extend it to poly-Genocchi version of as our recently published article [22].

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