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Research Article

The Generalized Difference Operator Δ_i^3 of Order Three and Its Domain in the Sequence Spaces ℓ_1 and bv

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Most recently, the generalized difference operator Δ_i^3 of order three was defined and its domain in Hahn sequence space h was calculated. In this paper, the spaces $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ are introduced as the domain of generalized difference operator Δ_i^3 of order three in the sequence spaces ℓ_1 and bv. Then, some topological properties of $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ are given, and some inclusion relations are shown. Additionally, algebraic dual, $\alpha-$, $\beta-$, and $\gamma-$ dual spaces of $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ are computed. In the last section, the classes $(\mu(\Delta_i^3): \lambda)$ and $(\lambda: \mu(\Delta_i^3))$ of matrix transformations are characterized, where $\mu = \{\ell_1, bv\}$ and $\lambda = \{c, c_0, \ell_1, \ell_\infty, bs, cs, bv, h\}$.

1. Preliminaries and Notations

The set of all complex valued sequences is denoted by ω and, according to the classification of ω each subset of ω, is called a sequence space. In the literature of sequences, the set ℓ_∞ which is called the set of all bounded sequences, the set c which is called the set of all convergent sequences, and the set c_0 which is called the set of all null sequences are called classical sequence spaces. If a sequence space μ is a complete metric space with continuous coordinates, then it is called FK-space. A normed FK-spaces is called a BK-space. Therefore, the classical sequence spaces ℓ_∞, c, and c_0 are BK-spaces with respect to the norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$. Moreover, the spaces of all absolutely $p-$summable, convergent series, null series, and bounded series are denoted by ℓ_p, cs, cs_0, and bs, respectively, where $1 < p < \infty$.

The space of absolutely summable sequences which is denoted by ℓ_1 and the space of all sequences of bounded variation which is denoted by bv are defined, respectively, as follows:

$$\ell_1 = \left\{ x = (x_k) \in \omega: \sum_{k=0}^{\infty} |x_k| < \infty \right\}, \quad (1)$$

and it is a BK-space with its norm $\|x\|_{\ell_1} = \sum_{k=0}^{\infty} |x_k| < \infty$:

$$bv = \left\{ x = (x_k) \in \omega: \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty \right\}, \quad (2)$$

and it is a BK-space with respect to the norm $\|x\|_{bv} = |x_0| + \sum_{k=1}^{\infty} |x_k - x_{k-1}| < \infty$. On the one hand, the sequence space bv is defined as the backward difference operator Δ domain on the sequence space ℓ_1, where $\Delta x_k = x_k - x_{k-1}$, for all $k \in \mathbb{N}$. On the other hand, we can also represent bv as

$$bv = \left\{ x = (x_k) \in \omega: \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty \right\}, \quad (3)$$

which is the forward difference operator Δ domain on the sequence space ℓ_1, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The space $bv_0 = bv \cap c_0$ and the inclusions $\ell_1 \subset bv_0 \subset bv \subset c$ are strictly held.

The alpha-dual λ^{α}, beta-dual λ^{β}, and gamma-dual λ^{γ} of a sequence space λ are defined by

$$\begin{aligned}\lambda^{\alpha} &:= \{x=(x_k)\in\omega: xy=(x_k y_k)\in \ell_1 \quad \text{for all } y=(y_k)\in\lambda\},\\ \lambda^{\beta} &:= \{x=(x_k)\in\omega: xy=(x_k y_k)\in cs \quad \text{for all } y=(y_k)\in\lambda\},\\ \lambda^{\gamma} &:= \{x=(x_k)\in\omega: xy=(x_k y_k)\in bs \quad \text{for all } y=(y_k)\in\lambda\}.\end{aligned} \tag{4}$$

The $\alpha-, \beta-, \gamma-$ duals of the spaces ℓ_1 and bv are defined by

$$\begin{aligned}&\{\ell_1\}^{\alpha}=\{\ell_1\}^{\beta}=\{\ell_1\}^{\gamma}=\ell_{\infty},\\ &\{bv\}^{\alpha}=\{bv_0\}^{\alpha}=\ell_1, \{bv\}^{\beta}=cs, \{bv_0\}^{\beta}\\ &\quad = bs, \{bv\}^{\gamma}=\{bv_0\}^{\gamma}=bs.\end{aligned} \tag{5}$$

Let $A=(a_{nk})_{k,n\in\mathbb{N}}$ be an infinite matrix and $\lambda,\mu\in\omega$. We write

$$y_k=(Ax)_n=\sum_k a_{nk}x_k, \tag{6}$$

and then, we say that A defines a matrix transformation from λ into μ as $A\colon \lambda\longrightarrow\mu$ if $Ax=\{(Ax)_n\}\in\mu$, for every $x\in\lambda$. We denote the set of all infinite matrices that map the sequence space λ into the sequence space μ by $(\lambda\colon\mu)$. Thus, $A\in(\lambda\colon\mu)$ if and only if the right side of (6) converges for every $n\in\mathbb{N}$, that is, $A_n\in\lambda^{\beta}$, for all $n\in\mathbb{N}$, and we have $Ax\in\mu$, for all $x\in\lambda$.

If a normed sequence space λ contains a sequence (b_n) with the following property that, for every $x\in\lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_{n\longrightarrow\infty}\|x-(\alpha_0 b_0+\alpha_1 b_1+\cdots+\alpha_n b_n)\|=0, \tag{7}$$

then (b_n) is called a Schauder basis for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x=\sum_k\alpha_k b_k$.

If λ is an FK-space, $\phi\subset\lambda$, and (e^k) is a basis for λ, then λ is said to have AK property, where e^k is a sequence whose only term in k^{th} place is 1; the others are zero, for each $k\in\mathbb{N}$ and $\phi=\text{span}\{e^k\}$. If ϕ is dense in λ, then λ is called AD-space; thus, AK implies AD.

Let λ be a sequence space and $A=(a_{nk})_{n,k\in\mathbb{N}}$ be an infinite matrix. Then, the matrix domain λ_A of an infinite matrix A in the sequence space λ is defined by

$$\lambda_A=\left\{x=(x_k)\in\omega\colon \left(\sum_k a_{nk}x_k\right)_{n\in\mathbb{N}} \text{existandisin}\,\lambda\right\}. \tag{8}$$

Wilansky (Theorem 4.4.2, p. 66 of [1]) defined that if λ is a sequence space, then the continuous dual λ_A^* of the space λ_A is given by

$$\lambda_A^*=\{f\colon f=g\circ A, g\in\lambda^*\}. \tag{9}$$

It is well known that $\ell_1^*=bv^*=\ell_{\infty}$ (see [2, 3]).

2. The New Difference Sequence Spaces $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$

Now, we define the new difference sequence spaces $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ as the domain of generalized difference matrix Δ_i^3 of order three in the sequence space ℓ_1 and bv. Then, we show that $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ are BK$-$ spaces and they are linearly isomorphic to the sequence spaces ℓ_1 and bv, respectively, and we show that Δ_i^3 is a linear and bounded operator over the sequence spaces ℓ_1 and bv to prove the inclusion relations among $\ell_1(\Delta_i^3)$ and ℓ_1 and $bv(\Delta_i^3)$ and bv, respectively.

The difference matrix Δ of order one was defined by Kizmaz [4] as $(\Delta x_k)=x_k-x_{k+1}$, and he studied its domain on classical sequence spaces. The generalized difference operator Δ^2 of order two was defined by Dutta and Baliarsing [5] as $(\Delta^2 x_k)=x_k-2x_{k-1}+x_{k-2}$, and they studied its spectrum on the sequence space c_0. Moreover, Baliarsing and Dutta [5] defined the generalized difference operator Δ_i^2 of order two as $(\Delta_i^2 x_k)=x_k-x_{k-1}+1/3x_{k-2}$ and studied its spectral subdivisions over the sequence spaces c_0 and ℓ_1. Then, again, Dutta and Baliarsing [6] defined generalized difference operator Δ^3 of order three as $(\Delta^3 x_k)=x_k-3x_{k-1}+3x_{k-2}-x_{k-3}$ and studied its spectrum over the sequence spaces c_0 and ℓ_1.

The generalized difference matrix $\Delta_i^3=(\delta_{nk})$ of order three was defined by Malkowsky et al. [7] as

$$\delta_{nk}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots\\ \frac{-3}{2} & 1 & 0 & 0 & 0 & 0 & \cdots\\ 1 & \frac{-3}{2} & 1 & 0 & 0 & 0 & \cdots\\ \frac{-1}{4} & 1 & \frac{-3}{2} & 1 & 0 & 0 & \cdots\\ 0 & \frac{-1}{4} & 1 & \frac{-3}{2} & 1 & 0 & \cdots\\ 0 & 0 & \frac{-1}{4} & 1 & \frac{-3}{2} & 1 & \cdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{10}$$

The matrix Δ_i^3 transforms a sequence x by

$$\left(\Delta_i^3 x\right)_k=\sum_{i=0}^{3}\frac{(-1)^i}{i+1}\binom{3}{i}x_{k-i}=x_k-\frac{3}{2}x_{k-1}+x_{k-2}-\frac{1}{4}x_{k-3}. \tag{11}$$

Then, most recently, the generalized difference matrix $\Delta_i^3=(\delta_{nk})$ domain in Hahn sequence space h was calculated and studied by Tug et al. [8].

Now, we define the following difference sequence spaces as the set of all sequences whose Δ_i^3-transforms are in the sequence spaces ℓ_1 and bv as follows:

$$\ell_1(\Delta_i^3) := \left\{ x = (x_k) \in \omega : \sum_{k=0}^{\infty} \left| (\Delta_i^3 x)_k \right| < \infty \right\},$$
$$bv(\Delta_i^3) := \left\{ x = (x_k) \in \omega : \sum_{k=0}^{\infty} \left| (\Delta_i^3 x)_k - (\Delta_i^3 x)_{k-1} \right| < \infty \right\}. \quad (12)$$

We define the sequence $y = (y_k)$ by the Δ_i^3- transform of the sequence $x = (x_k)$ as

$$y_k = (\Delta_i^3 x)_k = \sum_{i=0}^{3} \frac{(-1)^i}{i+1} \binom{3}{i} x_{k-i} = x_k - \frac{3}{2} x_{k-1} + x_{k-2} - \frac{1}{4} x_{k-3}, \quad (13)$$

for all $k \in \mathbb{N}$. The generalized difference matrix Δ_i^3 of order three is a triangle; then, it is invertable and the inverse is unique. Therefore, we obtain by considering the relation between the terms of $x = (x_k)$ and $y = (y_k)$ (13), and $x_0, x_{-1}, x_{-2}, \ldots$ are zero terms that

$$x_k = (\Delta_i^3)^{-1} y_k = y_k + \frac{3}{2} y_{k-1} - y_{k-2} + \frac{1}{4} y_{k-3}, \quad (14)$$

where $(\Delta_i^3)^{-1} = B = (b_{nk})$ is defined by

$$b_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{3}{2} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{5}{4} & \frac{3}{2} & 1 & 0 & 0 & 0 & \cdots \\ \frac{5}{8} & \frac{5}{4} & \frac{3}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{16} & \frac{5}{8} & \frac{5}{4} & \frac{3}{2} & 1 & 0 & \cdots \\ -\frac{7}{32} & \frac{1}{16} & \frac{5}{8} & \frac{5}{4} & \frac{3}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (15)$$

Theorem 1. *The sequence spaces $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ are BK-spaces with respect to the norm:*

$$\|x\|_{\ell_1(\Delta_i^3)} = \sum_{k=0}^{\infty} \left| (\Delta_i^3 x)_k \right|,$$
$$\|x\|_{bv(\Delta_i^3)} = |x_0| + \sum_{k=1}^{\infty} \left| (\Delta_i^3 x)_k - (\Delta_i^3 x)_{k-1} \right|, \quad (16)$$
or
$$\|x\|_{bv(\Delta_i^3)} = \sum_{k=1}^{\infty} \left| (\Delta_i^3 x)_k - (\Delta_i^3 x)_{k+1} \right|,$$

respectively.

Proof. Since ℓ_1 and bv are BK-spaces and Δ_i^3 is a triangle matrix, we obtain from the Theorem 4.3.2 of Wilansky (p. 61 of [1]) that $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$ are also BK-spaces. □

Relation (14) between the terms of the sequences $x = (x_k)$ and $y = (y_k)$ is given by the following calculation:

$$y_1 = x_1 \Longrightarrow x_1 = y_1$$
$$y_2 = x_2 - \frac{3}{2} x_1 \Longrightarrow x_2 = y_2 + \frac{3}{2} y_1$$
$$y_3 = x_3 - \frac{3}{2} x_2 + x_1 \Longrightarrow x_3 = y_3 + \frac{3}{2} y_2 + \frac{5}{4} y_1$$
$$\vdots$$
$$(\Delta_i^3 x)_k = y_k = x_k - \frac{3}{2} x_{k-1} + x_{k-2} - \frac{1}{4} x_{k-3} x_k = (\Delta_i^3)^{-1} y_k = y_k + \frac{3}{2} y_{k-1} - y_{k-2} + \frac{1}{4} y_{k-3}. \quad (17)$$

Thus, the following equation which was derived from (17) is given by

$$\lambda^3 - \frac{3}{2}\lambda^2 + \lambda - \frac{1}{4} = 0 \Longrightarrow 4\lambda^3 - 6\lambda^2 + 4\lambda - 1 = 0, \quad (18)$$

and we calculate roots of equation (18) as one real and two complex as follows:

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2} + i\frac{1}{2} \text{ and } \lambda_3 = \frac{1}{2} - i\frac{1}{2}. \quad (19)$$

Then, we have the following simple calculations by random three roots λ_1, λ_2, and λ_3 of equation (18):

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{3}{2}, \quad (20)$$

$$\lambda_1 \lambda_2 \lambda_3 = \frac{1}{4}, \quad (21)$$

$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 = 1, \quad (22)$$

$$\lambda_1^3 - \frac{3}{2}\lambda_1^2 + \lambda_1 - \frac{1}{4} = 0, \quad (23)$$

$$\lambda_1^2 + \lambda_2^2 - \frac{3}{2}(\lambda_1\lambda_2) + \lambda_1\lambda_2 + 1 = 0, \quad (24)$$

$$\begin{aligned}&\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ &- \frac{3}{2}(\lambda_1 + \lambda_2 + \lambda_3) + 1 = 0,\end{aligned} \quad (25)$$

$$\lambda_1 + \lambda_2 + \lambda_3 - \frac{3}{2} = 0. \quad (26)$$

Theorem 2. *The sequence space $\ell_1(\Delta_i^3)$ is linearly isomorphic to the sequence spaces ℓ_1, i.e., $\ell_1(\Delta_i^3) \cong \ell_1$.*

Proof. Suppose that the transformation T defined from the space $\ell_1(\Delta_i^3)$ onto ℓ_1 as $T: \ell_1(\Delta_i^3) \longrightarrow \ell_1$ by $x \longrightarrow y = Tx = \Delta_i^3 x$. The linearity of T is clear. Moreover, since $Tx = 0$ gives $x_k = 0$, for all $k \in \mathbb{N}$.

Let us take $y \in \ell_1$ and consider the sequence $x = (x_k)$ with respect to relation (14) as

$$x_k = \sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v}\lambda_2^v\lambda_3^i y_j, \quad (27)$$

for every $k \in \mathbb{N}$. By considering equations (23)–(26), we have the following Δ_i^3 transform of the sequence x as

$$\begin{aligned}\left(\Delta_i^3 x\right)_k &= x_k - \frac{3}{2}x_{k-1} + x_{k-2} - \frac{1}{4}x_{k-3} \\ &= \sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v}\lambda_2^v\lambda_3^i y_j - \frac{3}{2}\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1}\sum_{v=0}^{k-j-i-1} \lambda_1^{k-j-i-v-1}\lambda_2^v\lambda_3^i y_j + \sum_{j=0}^{k-2}\sum_{i=0}^{k-j-2}\sum_{v=0}^{k-j-i-2} \lambda_1^{k-j-i-v-2}\lambda_2^v\lambda_3^i y_j \\ &\quad - \frac{1}{4}\sum_{j=0}^{k-3}\sum_{i=0}^{k-j-3}\sum_{v=0}^{k-j-i-3} \lambda_1^{k-j-i-v-3}\lambda_2^v\lambda_3^i y_j \\ &= \sum_{j=0}^{k-3}\left[\sum_{i=0}^{k-j-3}\left[\sum_{v=0}^{k-j-i-3} \lambda_1^{k-j-i-v-3}\lambda_2^v\lambda_3^i\left(\lambda_1^3 - \frac{3}{2}\lambda_2^2 + \lambda_3 - \frac{1}{4}\right) + \lambda_3^i\lambda_2^{k-j-i}\left(\lambda_1^2 + \lambda_2^2 - \frac{3}{2}(\lambda_1\lambda_2) + \lambda_1\lambda_2 + 1\right)\right.\right. \\ &\quad \left.\left. + \lambda_3^{k-j}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 - \frac{3}{2}(\lambda_1 + t\lambda_2 n + q\lambda_3) + 1\right)\right]\right] y_j \\ &\quad + \left[y_{k-2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 - \frac{3}{2}(\lambda_1 + t\lambda_2 n + q\lambda_3) + 1\right) + y_{k-1}\left(\lambda_1 + \lambda_2 + \lambda_3 - \frac{3}{2}\right) + y_k\right] \\ &= y_k,\end{aligned} \quad (28)$$

for all $k \in \mathbb{N}$. Thus, $Tx = y$, for all $x = (x_k) \in \ell_1(\Delta_i^3)$; then, T is surjective. Moreover, for every $x = (x_k) \in \ell_1(\Delta_i^3)$, we have

$$\|x\|_{\ell_1(\Delta_i^3)} = \sum_{k=0}^{\infty}\left|\left(\Delta_i^3 x\right)_k\right| = \sum_{k=0}^{\infty}|y_k| = \|y\|_{\ell_1}. \quad (29)$$

Hence, x is an element of $\ell_1(\Delta_i^3)$, and clearly, T is surjective and preserves the norm. Therefore, $\ell_1(\Delta_i^3) \cong \ell_1$. It completes the proof. □

Theorem 3. *The sequence space $bv(\Delta_i^3)$ is linearly isomorphic to the sequence space bv, i.e., $bv(\Delta_i^3) \cong bv$.*

Proof. Suppose that the transformation T is defined from the space $bv(\Delta_i^3)$ onto bv as $T: bv(\Delta_i^3) \longrightarrow bv$ by $x \longrightarrow y = Tx = \Delta_i^3 x$. The linearity of T is clear. Moreover, since $Tx = 0$ gives $x_k = 0$, for all $k \in \mathbb{N}$, the rest of the proof can be followed by equation (28) in the proof of Theorem 2, and thus, $Tx = y$, for all $x = (x_k) \in bv(\Delta_i^3)$; then, T is surjective. Moreover, for every $x = (x_k) \in bv(\Delta_i^3)$, we have

$$\begin{aligned}\|x\|_{bv(\Delta_i^3)} &= |x_0| + \sum_{k=1}^{\infty}\left|\left(\Delta_i^3 x\right)_k - \left(\Delta_i^3 x\right)_{k-1}\right| \\ &= |y_0| + \sum_{k=1}^{\infty}|y_k - y_{k-1}| = \|y\|_{bv}.\end{aligned} \quad (30)$$

Hence, x is an element of $bv(\Delta_i^3)$, and clearly, T is surjective and preserves the norm. Therefore, $bv(\Delta_i^3) \cong bv$. It completes the proof. □

Theorem 4. *The inclusion $\ell_1(\Delta_i^3) \subset bv(\Delta_i^3)$ strictly holds.*

Proof. Suppose that $x \in \ell_1(\Delta_i^3)$, then $y = \Delta_i^3 x \in \ell_1$. Since $\ell_1 \subset bv$, then $y = \Delta_i^3 x \in bv$ and it says $x \in bv(\Delta_i^3)$ and it shows the inclusion $\ell_1(\Delta_i^3) \subset bv(\Delta_i^3))$ holds. To show that the inclusion $\ell_1(\Delta_i^3) \subset bv(\Delta_i^3)$ is strict, let us define $(x_k) = e = (1, 1, \ldots)$. Then, clearly, $x \in bv(\Delta_i^3)$, but $x \notin \ell_1(\Delta_i^3)$, since $\Delta_i^3 x_k \longrightarrow 1/4 \nrightarrow 0$ as $(k \longrightarrow \infty)$. This completes the proof. □

Lemma 1. *The matrix $A = (a_{nk})$ is a bounded linear operator, $A \in B(\ell_1)$, from ℓ_1 to itself, if and only if the supremum of ℓ_1 norms of the columns of A is bounded, i.e.,*

$$\sup_{k\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{nk}| < \infty. \tag{31}$$

Theorem 5. *$\Delta_i^3 \colon \ell_1 \longrightarrow \ell_1$ is a bounded linear operator.*

Proof. The linearity is clear. We should show that $\Delta_i^3 \in B(\ell_1)$ which means that Δ_i^3 satisfies the conditions of Lemma 1 with δ_{nk} instead of a_{nk}, that is,

$$\|\Delta_i^3\|_{(\ell_1;\ell_1)} = \frac{15}{4}. \tag{32}$$

This completes the proof. □

Lemma 2. *The matrix $A = (a_{nk})$ is a bounded linear operator, $A \in B(bv)$, from bv to itself, if and only if*

$$\sup_k \sum_{n=0}^{\infty} \left| \sum_{i=k}^{\infty} \left(a_{ni} - a_{n-1,i}\right) \right| < \infty. \tag{33}$$

Theorem 6. *$\Delta_i^3 \colon bv \longrightarrow bv$ is a bounded linear operator.*

Proof. The linearity is clear. We should show that $\Delta_i^3 \in B(bv)$ which means that Δ_i^3 satisfies the conditions of Lemma 2 with δ_{nk} instead of a_{nk}, that is,

$$\|\Delta_i^3\|_{(bv;bv)} = \sup_k \sum_{n=0}^{\infty} \left| \sum_{i=k}^{\infty} \left(\delta_{ni} - \delta_{n-1,i}\right) \right| = \frac{15}{4}. \tag{34}$$

This completes the proof. □

Theorem 7. *$\ell_1 = \ell_1(\Delta_i^3)$.*

Proof. Since the operator Δ_i^3 is a bounded and linear operator on the sequence space ℓ_1 by Theorem 5, $\Delta_i^3 \in (\ell_1, \ell_1)$ if and only if condition (31) is satisfied. Moreover, $\Delta_i^3 x \in \ell_1$, for every $x \in \ell_1$. This shows that the inclusion $\ell_1 \subset \ell_1(\Delta_i^3)$ holds.

Moreover, the matrix $B = (b_{nk})$, the inverse matrix of Δ_i^3, which can be reduced from the inverse matrix $B = (b_{nk})$ in the Theorem 2 of [2] by only choosing $\lambda = 0$. Then, we write the following to calculate $(\Delta_i^3)^{-1} \in (\ell_1; \ell_1)$. Therefore, the operator $(\Delta_i^3)^{-1}$ has the following equation and the following calculations of its roots:

$$\left(\Delta_i^3\right)^{-1} y_k = y_k + \frac{3}{2} y_{k-1} - y_{k-2} + \frac{1}{4} y_{k-3}, \tag{35}$$

where $(\Delta_i^3)^{-1} = B = (b_{nk})$ is defined by

$$b_{nk} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{3}{2} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{5}{4} & \frac{3}{2} & 1 & 0 & 0 & 0 & \cdots \\ \frac{5}{8} & \frac{5}{4} & \frac{3}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{16} & \frac{5}{8} & \frac{5}{4} & \frac{3}{2} & 1 & 0 & \cdots \\ -\frac{7}{32} & \frac{1}{16} & \frac{5}{8} & \frac{5}{4} & \frac{3}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{36}$$

The notation $(\Delta_i^3)^{-1} y_k = y_k + 3/2 y_{k-1} - y_{k-2} + 1/4 y_{k-3}$ gives us the following equation as

$$w^3 + \frac{3}{2} w^2 - w + \frac{1}{4} = 0 \Longrightarrow 4w^3 + 6w^2 - 4w + 1 = 0. \tag{37}$$

Then, one real and two complex roots of equation (37) are

$$\begin{aligned} w_1 &= -\frac{1}{2} - \frac{1}{2}\left(\frac{\sqrt[3]{36 - \sqrt{267}}}{3^{2/3}} + \frac{7}{\sqrt[3]{3\left(36 - \sqrt{267}\right)}} \right), \\ w_2 &= -\frac{1}{2} + \frac{1}{4}\left(\frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{36 - \sqrt{267}}}{3^{2/3}} + \frac{7\left(1 - i\sqrt{3}\right)}{\sqrt[3]{3\left(36 - \sqrt{267}\right)}} \right), \\ w_3 &= -\frac{1}{2} + \frac{1}{4}\left(\frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{36 - \sqrt{267}}}{3^{2/3}} + \frac{7\left(1 + i\sqrt{3}\right)}{\sqrt[3]{3\left(36 - \sqrt{267}\right)}} \right). \end{aligned} \tag{38}$$

Then, we have the following simple calculations by random three roots w_1, w_2, and w_3 of equation (37):

$$\begin{aligned}
w_1 + w_2 + w_3 &= -\frac{3}{2},\\
w_1 w_2 w_3 &= -\frac{1}{4},\\
w_1 w_2 + w_2 w_3 + w_1 w_3 &= -1,\\
w_1^3 + \frac{3}{2} w_1^2 - w_1 + \frac{1}{4} &= 0,\\
w_1^2 + w_2^2 + \frac{3}{2}\left(w_1 + w_2\right) + w_1 w_2 - 1 &= 0,\\
w_1^2 + w_2^2 + w_3^2 + w_1 w_2 + w_2 w_3 + w_1 w_3 + \frac{3}{2}\left(w_1 + w_2 + w_3\right) - 1 &= 0,\\
w_1 + w_2 + w_3 + \frac{3}{2} &= 0,
\end{aligned} \tag{39}$$

$$b_{nk} = \begin{cases} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} w_1^{k-j-i-v} w_2^{v} w_3^{i}, & 0 \le k \le n,\\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. We can show that $B \in (\ell_1, \ell_1)$, and if $x \in \ell_1(\Delta_i^3)$, then $y = \Delta_i^3 x \in \ell_1$ and $x = (\Delta_i^3)^{-1} y = By \in \ell_1$. Thus, we can say that $\ell_1(\Delta_i^3) \subset \ell_1$. This completes the proof. □

Theorem 8. *$bv = bv(\Delta_i^3)$.*

Proof. Since the proof can be done similarly as in the proof of Theorem 7, we omit it. □

Theorem 9. *Let $\alpha_k = (\Delta_i^3 x)_k$, for all $k \in \mathbb{N}$ and $\mu = \{\ell_1, bv\}$. Define the sequence $\{u^{(k)}\} = \{u_n^{(k)}\}_{n\in\mathbb{N}}$ in the sequence space $\mu(\Delta_i^3)$ as follows:*

$$u_n^{(k)} = \begin{cases} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^{v} \lambda_3^{i}, & 0 \le k \le n,\\ 0, & k > n, \end{cases} \tag{40}$$

for every fixed $k \in \mathbb{N}$. Then, $\{u^{(k)}\} = \{u_n^{(k)}\}_{n\in\mathbb{N}}$ is a basis for $\mu(\Delta_i^3)$, and there is a unique representation of $x \in \mu(\Delta_i^3)$ as

$$x = \sum_k \alpha_k u^{(k)}. \tag{41}$$

Proof. Since the proof can be done similarly for both sequence space, we consider to prove for the sequence space ℓ_1. First, we need to show that $\{u^{(k)}\} \in \ell_1(\Delta_i^3)$, and it is enough to show $\Delta_i^3 u^{(k)} \in h$, for all $k \in \mathbb{N}$. To show this

$$\begin{aligned}
\Delta_i^3 u^{(k)} &= u^{(k)} - \frac{3}{2}u^{(k-1)} + u^{(k-2)} - \frac{1}{4}u^{(k-3)} \\
&= \sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\lambda_1^{k-j-i-v}\lambda_2^{v}\lambda_3^{i} - \frac{3}{2}\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1}\sum_{v=0}^{k-j-i-1}\lambda_1^{k-j-i-v-1}\lambda_2^{v}\lambda_3^{i} \\
&\quad + \sum_{j=0}^{k-2}\sum_{i=0}^{k-j-2}\sum_{v=0}^{k-j-i-2}\lambda_1^{k-j-i-v-2}\lambda_2^{v}\lambda_3^{i} - \frac{1}{4}\sum_{j=0}^{k-3}\sum_{i=0}^{k-j-3}\sum_{v=0}^{k-j-i-3}\lambda_1^{k-j-i-v-3}\lambda_2^{v}\lambda_3^{i} \\
&= \sum_{j=0}^{k-3}\Bigg[\sum_{i=0}^{k-j-3}\Bigg[\sum_{v=0}^{k-j-i-3}\lambda_1^{k-j-i-v-3}\lambda_2^{v}\lambda_3^{i}\left(\lambda_1^3 - \frac{3}{2}\lambda_1^2 + \lambda_1 - \frac{1}{4}\right) + \lambda_3^{i}\lambda_2^{k-j-i}\left(\lambda_1^2 + \lambda_2^2 - \frac{3}{2}(\lambda_1+\lambda_2) + \lambda_1\lambda_2 + 1\right) \\
&\quad + \lambda_3^{k-j}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 - \frac{3}{2}(\lambda_1+\lambda_2+\lambda_3) + 1\right)\Bigg]\Bigg] \\
&\quad + \left[\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 - \frac{3}{2}(\lambda_1+\lambda_2+\lambda_3) + 1\right) + \left(\lambda_1 + \lambda_2 + \lambda_3 - \frac{3}{2}\right) + 1\right] = 1,
\end{aligned} \tag{42}$$

for every $k \in \mathbb{N}$. Then, clearly, one can see that $\Delta_i^3 u^{(k)} = e^k \in \ell_1$ and then $u^{(k)} \in \ell_1(\Delta_i^3)$.

Let us take a sequence $x \in \ell_1(\Delta_i^3)$. Then, we obtain the following representation for every nonnegative integer m as

$$x^{[m]} = \sum_{k}^{m} \alpha_k u^{(k)}. \tag{43}$$

Then, the following holds by taking the Δ_i^3 transform of (43) that

$$\Delta_i^3 x^{[m]} = \sum_{k}^{m} \alpha_k \Delta_i^3 u^{(k)} = \sum_{k}^{m} \left(\Delta_i^3 x\right)_k e^k, \tag{44}$$

and from (44), we have

$$\left\{\Delta_i^3\left(x - x^{[m]}\right)\right\}_n = \begin{cases} 0, & 0 \le n \le m, \\ \left(\Delta_i^3 x\right)_n, & n > m. \end{cases} \tag{45}$$

Therefore, for every given $\varepsilon > 0$, there exists an integer m_0 such that

$$\sum_{n=m}^{\infty}\left|\left(\Delta_i^3 x\right)_n\right| < \frac{\varepsilon}{2}, \tag{46}$$

for all $m \ge m_0$. Hence,

$$\left\|x - x^{[m]}\right\|_{\ell_1(\Delta_i^3)} = \sum_{n=m}^{\infty}\left|\left(\Delta_i^3 x\right)_n\right| \le \sum_{n=m_0}^{\infty}\left|\left(\Delta_i^3 x\right)_n\right| < \frac{\varepsilon}{2}, \tag{47}$$

for all $m \ge m_0$. This proves that $x \in \ell_1(\Delta_i^3)$. □

3. Dual Spaces of the Sequence Spaces $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$

We begin this section by calculating the algebraic dual space of $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$, respectively.

Theorem 10. *The algebraic dual $\{\ell_1(\Delta_i^3)\}^*$ of the space $\ell_1(\Delta_i^3)$ is the sequence space ℓ_∞.*

Proof. Let us define $T: \{\ell_1(\Delta_i^3)\}^* \longrightarrow \ell_\infty$ with $T(f) = f(u^{(k)})$ which is a surjective linear map, and T is injective since $u^{(k)}$ is a basis for $\ell_1(\Delta_i^3)$. Let $f \in \{\ell_1(\Delta_i^3)\}^*$, and since $x \in \ell_1(\Delta_i^3)$, we can write

$$f(x) = f\left(\sum_{k=0}^{\infty}\left(\Delta_i^3 x_k\right)u^{(k)}\right) = \sum_{k=0}^{\infty}\left(\Delta_i^3 x_k\right)f\left(u^{(k)}\right). \tag{48}$$

Then, we have

$$\begin{aligned}
|f(x)| &= \left|f\left(\sum_{k=0}^{\infty}\left(\Delta_i^3 x_k\right)u^{(k)}\right)\right| \\
&= \left|\sum_{k=0}^{\infty}\left(\Delta_i^3 x_k\right)f\left(u^{(k)}\right)\right| \\
&\le \sum_{k=0}^{\infty}\left|\left(\Delta_i^3 x_k\right)\right|\left|f\left(u^{(k)}\right)\right| \\
&\le \sup_{k\ge1}\left|f\left(u^{(k)}\right)\right|\sum_{k=0}^{\infty}\left|\left(\Delta_i^3 x_k\right)\right| = \|T(f)\|_\infty \|x\|_{\ell_1(\Delta_i^3)}.
\end{aligned} \tag{49}$$

Thus, $\|f\| \le \|T(f)\|_\infty$.

Moreover, since $|f(u^{(k)})| \le \|f\|\|u^{(k)}\| = \|f\|$, for every $k \in \mathbb{N}$, then $\|T(f)\|_\infty = \sup_{k\ge1}|f(u^{(k)})| = \|f\|$. Therefore, $\|f\| = \|T(f)\|_\infty$. □

Theorem 11. *The algebraic dual $\{bv(\Delta_i^3)\}^*$ of the space $bv(\Delta_i^3)$ is the sequence space ℓ_∞.*

Proof. Since the proof can be followed similarly as in Theorem 10, we omit the repetition. □

Now, we start with the following lemma which is needed in proving our theorems. Here and after, we denote the collection of all finite subsets of $\mathbb{N}$ by $\mathscr{K}$.

Lemma 3 (see [9]). *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (\ell_1 : \ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty. \tag{50}$$

(i) $A \in (\ell_1 : \ell_1)$ if and only if

$$\sup_{k\in\mathbb{N}} \sum_n |a_{nk}| < \infty. \tag{51}$$

(iii) $A \in (\ell_1 : c)$ if and only if (50) holds, and

$$\exists a_k \in \mathbb{C} \text{ such that } \lim_{n\longrightarrow\infty} a_{nk} = a_k \quad \text{forall } k \in \mathbb{N}. \tag{52}$$

(iv) $A \in (c : \ell_1)$ if and only if

$$\sup_{\mathcal{K}\in\mathbb{N}\text{finite}} \sum_n \left| \sum_{k\in\mathcal{K}} a_{nk} \right| < \infty. \tag{53}$$

Lemma 4 (see [9]). *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold:*

(i) $A \in (bv : \ell_1)$ if and only if

$$\sup_{l\in\mathbb{N}} \sum_n \left| \sum_{k=l}^{\infty} a_{nk} \right| < \infty. \tag{54}$$

(ii) $A \in (bv : bs)$ if and only if

$$\sup_{m,l\in\mathbb{N}} \left| \sum_{n=0}^{m} \sum_{k=l}^{\infty} a_{nk} \right| < \infty. \tag{55}$$

(iii) $A \in (bv : cs)$ if and only if (55) holds, and

$$\sum_n a_{nk} \text{ converges for each } k \in \mathbb{N}, \tag{56}$$

$$\sum_n \sum_k a_{nk} \text{converges.} \tag{57}$$

(iv) $A \in (bv : bv)$ if and only if

$$\sup_k \sum_{n=0}^{\infty} \left| \sum_{i=k}^{\infty} \left(a_{ni} - a_{n-1,i} \right) \right| < \infty. \tag{58}$$

(v) $A \in (bv : \ell_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}} \left| \sum_{i=k}^{\infty} a_{ni} \right| < \infty. \tag{59}$$

(vi) $A \in (bv : c)$ if and only if (57) and (59) hold and

$$Ae \in c. \tag{60}$$

Theorem 12. *Let the set d_1 be defined as*

$$d_1 = \left\{ a = (a_k) \in \omega : \sup_{j\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k \right| < \infty \right\}. \tag{61}$$

Then, $\{\ell_1(\Delta_i^3)\}^\alpha = d_1$.

Proof. Let the sequence $a = (a_k)$ be in ω. By relation (27), we have the following equality:

$$a_k x_k = \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k y_j = (Ay)_k, \tag{62}$$

where the matrix $A = (a_{kj})$ is defined via the sequence $a = (a_k)$ as

$$a_{kj} = \begin{cases} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k, & 1 \le j \le k, \\ 0, & j > k. \end{cases} \tag{63}$$

Now, clearly, we can say that $ax = (a_k x_k) \in \ell_1$ whenever $x = (x_k) \in \ell_1(\Delta_i^3)$ if and only if $Ay \in \ell_1$ whenever $y = (y_k) \in \ell_1$. Therefore, $A \in (\ell_1; \ell_1)$ and the condition of Lemma 3 (ii) holds, that is,

$$\sup_{j\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k \right| < \infty. \tag{64}$$

It completes the proof. □

Theorem 13. *$\{\ell_1(\Delta_i^3)\}^\beta = d_2 \cap d_3$, where*

$$d_2 = \left\{ a = (a_k) \in \omega : \sup_{n,k\in\mathbb{N}} \left| \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right| < \infty \right\}, \tag{65}$$

$$d_3 = \left\{ a = (a_k) \in \omega : \lim_{n\longrightarrow\infty} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \text{ exists for each } k = 1, 2, 3, \ldots \right\}. \tag{66}$$

Proof. Suppose that $a = (a_k) \in \omega$. Then, let us consider the following equality:

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i y_j$$
$$\sum_{k=0}^{n} \left(\sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right) y_k = (By)_n, \tag{67}$$

for every $n \in \mathbb{N}$, where we define the matrix $B = (b_{nk})$ as

$$b_{nk} = \begin{cases} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j, & 1 \le k \le n, \\ 0, & k > n, \end{cases} \tag{68}$$

for all $k, n \in \mathbb{N}$. By the definition $\beta-$ dual of a sequence space, we can say that $ax = (a_k x_k) \in cs$ wherever $x = (x_k) \in \ell_1(\Delta_i^3)$ if and only if $By \in c$ wherever $y = (y_k) \in \ell_1$. Thus, $B \in (\ell_1; c)$, and the conditions of Lemma 3 (iii) hold. That is,

$$\sup_{n,k\in\mathbb{N}} \left| \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right|$$
$$< \infty \lim_{n\longrightarrow\infty} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \text{ exists for each } k = 1, 2, 3, \ldots. \tag{69}$$

It completes the proof. □

Theorem 14. *The gamma-dual* $\{\ell_1(\Delta_i^3)\}^\gamma$ *of the sequence space* $\ell_1(\Delta_i^3)$ *is the set* d_2.

Proof. The proof can be easily done by considering the similar path in the proof of Theorem 13. We only state here that $ax = (a_k x_k) \in bs$ wherever $x = (x_k) \in \ell_1(\Delta_i^3)$ if and only if $By \in \ell_\infty$ wherever $y = (y_k) \in \ell_1$. Thus, $B \in (\ell_1; \ell_\infty)$, and the condition of the Lemma 3 (i) holds. That is,

$$\sup_{n,k\in\mathbb{N}} \left| \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right| < \infty. \tag{70}$$

It completes the proof. □

Theorem 15. *Let the set* S_1 *be defined as*

$$S_1 = \left\{ a = (a_k) \in \omega \colon \sup_{l\in\mathbb{N}} \sum_{n=1}^{\infty} \left| \sum_{k=l}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k \right| < \infty \right\}. \tag{71}$$

Then, $\{bv(\Delta_i^3)\}^\alpha = S_1$.

Proof. Let the sequence $a = (a_k)$ be in ω. By relation (27), we have the following equality:

$$a_k x_k = \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k y_j = (Ay)_k, \tag{72}$$

where the matrix $A = (a_{kj})$ is defined via the sequence $a = (a_k)$ as

$$a_{kj} = \begin{cases} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k, & 1 \le j \le k, \\ 0, & j > k. \end{cases} \tag{73}$$

Now, clearly, we can say that $ax = (a_k x_k) \in \ell_1$ whenever $x = (x_k) \in bv(\Delta_i^3)$ if and only if $Ay \in \ell_1$ whenever $y = (y_k) \in bv$. Therefore, $A \in (bv; \ell_1)$ and the condition of Lemma 4 (i) holds, that is,

$$\sup_{l\in\mathbb{N}} \sum_{n=1}^{\infty} \left| \sum_{k=l}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_k \right| < \infty. \tag{74}$$

It completes the proof. □

Theorem 16. $\{bv(\Delta_i^3)\}^\beta = S_2 \cap S_3 \cap S_4$, *where*

$$S_2 = \left\{ a = (a_k) \in \omega : \sum_{n} \sum_{j} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \text{ converges} \right\},$$

$$S_3 = \left\{ a = (a_k) \in \omega : \sup_{n,v\in\mathbb{N}} \left| \sum_{k=v}^{\infty} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right| < \infty \right\},$$

$$S_4 = \left\{ a = (a_k) \in \omega : \lim_{n \longrightarrow \infty} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \text{exists for each } k = 1, 2, 3, \ldots \right\}. \quad (75)$$

Proof. Suppose that $a = (a_k) \in \omega$. Then, let us consider the following equality:

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i y_j, \\ &= \sum_{k=0}^{n} \left(\sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right) y_k = (By)_n, \end{aligned} \quad (76)$$

for every $n \in \mathbb{N}$, where we define the matrix $B = (b_{nk})$ as

$$b_{nk} = \begin{cases} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j, & 1 \le k \le n, \\ 0, & k > n, \end{cases} \quad (77)$$

for all $k, n \in \mathbb{N}$. By the definition $\beta-$ dual of a sequence space, we can say that $ax = (a_k x_k) \in cs$ wherever $x = (x_k) \in bv(\Delta_i^3)$ if and only if $By \in c$ wherever $y = (y_k) \in bv$. Thus, $B \in (bv; c)$ and the conditions of Lemma 4 (vi) hold. That is,

$$\begin{aligned} &\sum_n \sum_j \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \text{ converges}, \\ &\sup_{m,n \in \mathbb{N}} \sum_n \left| \sum_{k=0}^{m} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right| \\ &< \infty, \lim_{n \longrightarrow \infty} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \text{ exists for each } k = 1, 2, 3, \ldots. \end{aligned} \quad (78)$$

It completes the proof. □

Theorem 17. *The gamma-dual $\{bv(\Delta_i^3)\}^\gamma$ of the sequence space $\{bv(\Delta_i^3)\}$ is the set S_3.*

Proof. The proof can be easily done by considering the similar path in the proof of Theorem 16. We only state here that $ax = (a_k x_k) \in bs$ wherever $x = (x_k) \in bv(\Delta_i^3)$ if and only if $By \in \ell_\infty$ wherever $y = (y_k) \in bv$. Thus, $B \in (bv; \ell_\infty)$ and the condition of Lemma 4 (v) holds. That is,

$$\sup_{n,v \in \mathbb{N}} \left| \sum_{k=v}^{\infty} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_j \right| < \infty. \quad (79)$$

It completes the proof. □

4. Matrix Transformations on $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$

In this section, we characterize the matrix classes $(\lambda(\Delta_i^3); \mu)$ and $(\mu; \lambda(\Delta_i^3))$, where $\lambda \in \{\ell_1, bv\}$ and μ represents any sequence space. Throughout this section, we define the following matrices $E = (e_{nk})$ and $D = (d_{nk})$ as

$$\begin{aligned} e_{nk} &= \sum_{j=k}^{\infty} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \lambda_1^{j-k-i-v} \lambda_2^j \lambda_3^i a_{nj}, \\ d_{nk} &= a_{nk} - \frac{3}{2} a_{n-1,k} + a_{n-2,k} - \frac{1}{4} a_{n-3,k}, \end{aligned} \quad (80)$$

for all $k, n \in \mathbb{N}$.

Theorem 18. *The infinite matrix $A \in (\lambda(\Delta_i^3); \mu)$ if and only if*

$$A_n \in \left\{ \lambda\left(\Delta_i^3\right) \right\}^\beta, \quad (81)$$

$$E \in (\lambda; \mu). \quad (82)$$

Proof. Suppose that $A \in (\lambda(\Delta_i^3); \mu)$. Then, Ax exists and is in the sequence space μ for every arbitrary sequence $x = (x_k) \in \lambda(\Delta_i^3)$, that is, $y = \Delta_i^3 x \in \lambda$. Then, clearly $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\lambda(\Delta_i^3)\}^\beta$ holds. Since Ax exists for every arbitrary sequence $x = (x_k) \in \lambda(\Delta_i^3)$, we consider the following equality derived from the m^{th} partial sum of the series $\sum_k a_{nk} x_k$:

$$\begin{aligned} \sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \left(\sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i y_j \right) \\ &= \sum_{k=0}^{m} \left(\sum_{j=k}^{m} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \lambda_1^{k-j-i-v} \lambda_2^j \lambda_3^i a_{nj} \right) y_k. \end{aligned} \quad (83)$$

Thus, by letting $m \longrightarrow \infty$ in equality (83), we can see that

$$\sum_k a_{nk} x_k = \sum_k e_{nk} y_k, \tag{84}$$

for every $n \in \mathbb{N}$. Therefore, $Ax = Ey$, for every $y \in \lambda$, that is, $E \in (\lambda; \mu)$.

Conversely, assume that conditions (81) and (82) hold and suppose that $x = (x_k) \in \lambda(\Delta_i^3)$. We can clearly say by (81) that Ax exists, and since the sequence spaces $\lambda(\Delta_i^3)$ and h are isomorphic, then we have $\Delta_i^3 x = y \in h$. We should show that $Ax \in \mu$ for every $x = (x_k) \in \lambda(\Delta_i^3)$. It can be easily seen by applying equation (83) that $Ax = Ey$, and since (82) holds, we can state here that $A \in (\lambda(\Delta_i^3); \mu)$ which complete the proof. □

Corollary 1. *The following statements hold for the infinite matrix $A = (a_{nk})$:*

(i) $A \in (\ell_1(\Delta_i^3); \ell_\infty)$ if and only if (50) holds with e_{nk} instead of a_{nk}

(ii) $A \in (\ell_1(\Delta_i^3); \ell_1)$ if and only if (51) holds with e_{nk} instead of a_{nk}

(iii) $A \in (\ell_1(\Delta_i^3); c)$ if and only if (50) and (52) hold with e_{nk} instead of a_{nk}

Corollary 2. *The following statements hold for the infinite matrix $A = (a_{nk})$:*

(i) $A \in (bv(\Delta_i^3); \ell_1)$ if and only if (54) holds with e_{nk} instead of a_{nk}

(ii) $A \in (bv(\Delta_i^3); bs)$ if and only if (55) holds with e_{nk} instead of a_{nk}

(iii) $A \in (bv(\Delta_i^3); cs)$ if and only if (55)–(57) hold with e_{nk} instead of a_{nk}

(iv) $A \in (bv(\Delta_i^3); bv)$ if and only if (58) holds with e_{nk} instead of a_{nk}

(v) $A \in (bv(\Delta_i^3); \ell_\infty)$ if and only if (59) holds with e_{nk} instead of a_{nk}

(vi) $A \in (bv(\Delta_i^3); c)$ if and only if (57), (59), and (60) hold with e_{nk} instead of a_{nk}

Lemma 5 (see Corollary 5 of [3]). *The infinite matrix $A = (a_{nk}) \in (bv; h)$ if and only if*

$$\lim_{n \longrightarrow \infty} \sum_{k=1}^{m} a_{nk} = 0, \tag{85}$$

$$\sup_{k \in \mathbb{N}} \sum_{n=1}^{\infty} n \left| \sum_{k=1}^{m} \left(a_{nk} - a_{n+1,k} \right) \right| < \infty, \tag{86}$$

$$Ae \in h. \tag{87}$$

Lemma 6 (see Theorem 4.15 of [3]). *We have $A \in (\ell_1, h)$ if and only if holds and*

$$\lim_{n \longrightarrow \infty} a_{nk} = 0, \quad (k = 1, 2, \ldots), \tag{88}$$

$$\|A\|_{(\ell_1, h)} = \sup_k \sum_{n=1}^{\infty} n \left| a_{nk} - a_{n+1,k} \right| < \infty. \tag{89}$$

Corollary 3. *The following statements hold for the infinite matrix $A = (a_{nk})$.*

(i) $A \in (bv(\Delta_i^3), h)$ if and only if the conditions in (85)–(87) hold with e_{nk} instead of a_{nk}

(ii) $A \in (\ell_1(\Delta_i^3), h)$ if and only if the conditions in (88) and (89) hold with e_{nk} instead of a_{nk}

Theorem 19. *The infinite matrix $A \in (\mu; \lambda(\Delta_i^3))$ if and only if*

$$D \in (\mu; \lambda). \tag{90}$$

Proof. Suppose that $A \in (\mu; \lambda(\Delta_i^3))$. Then, Ax exists, and it is in $\lambda(\Delta_i^3)$, that is, $\Delta_i^3(Ax) \in \lambda$. Then, we observe the following equality that

$$\begin{aligned} \left\{ \Delta_i^3 (Ax) \right\}_n &= (Ax)_n - \frac{3}{2}(Ax)_{n-1} + (Ax)_{n-2} - \frac{1}{4}(Ax)_{n-3} \\ &= \sum_k \left(a_{nk} - \frac{3}{2} a_{n-1,k} + a_{n-2,k} - \frac{1}{4} a_{n-3,k} \right) x_k \\ &= (Dx)_n. \end{aligned} \tag{91}$$

Thus, it shows $Dx \in \lambda$, for every $x \in \mu$, i.e., $D \in (\mu; \lambda)$. This completes the proof. □

Corollary 4. *The following statements hold for the infinite matrix $A = (a_{nk})$.*

(i) $A \in (\ell_1, \ell_1(\Delta_i^3))$ if and only if (51) holds with d_{nk} instead of a_{nk}

(ii) $A \in (c, \ell_1(\Delta_i^3))$ if and only if (53) holds with d_{nk} instead of a_{nk}

(iii) $A \in (bv, bv(\Delta_i^3))$ if and only if (58) holds with d_{nk} instead of a_{nk}

Lemma 7 (see [10]). *The infinite matrix $A \in (h{:}\ \ell_1)$ if and only if*

$$\sum_{n=1}^{\infty} \left| a_{nk} \right| \text{converges}, \quad (k = 1, 2, \ldots), \tag{92}$$

$$\sup_m \frac{1}{m} \sum_{n=1}^{\infty} \left| \sum_{k=1}^{m} a_{nk} \right| < \infty. \tag{93}$$

Lemma 8 (see Theorem 4.3 of [3]). *The infinite matrix $A = (a_{nk}) \in (h; bv)$ if and only if*

$$\sup_{m\in\mathbb{N}} \frac{1}{m} \sum_{n} \left| \sum_{k=1}^{m} \left(a_{nk} - a_{n-1,k} \right) \right| < \infty. \qquad (94)$$

Corollary 5. *The following statements hold for the infinite matrix* $A = (a_{nk})$*:*

(i) $A \in (h, \ell_1(\Delta_i^3))$ *if and only if the conditions in (92) and (93) hold with* d_{nk} *instead of* a_{nk}

(ii) $A \in (h, bv(\Delta_i^3))$ *if and only if the condition in (94) holds with* d_{nk} *instead of* a_{nk}

5. Conclusion

The difference operator and generalized difference operator of several orders were studied by several distinguished mathematicians (see [11–26]) as a matrix domain on several sequence spaces. The generalized difference operator Δ_i^3 of order three was defined and the spectrum of Δ_i^3 on the Hahn sequence space h calculated by Malkowsky et al. [7]. Then, the matrix domain of Δ_i^3 in Hahn sequence space h was calculated by Tug et al. [8].

In this research paper, we calculated the generalized difference operator Δ_i^3 of order three domain in the sequence spaces ℓ_1 and bv. Then, we stated some topological properties of $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$, and we showed some inclusion relations. Moreover, we calculated the algebraic dual, $\alpha-$, $\beta-$, and $\gamma-$ dual spaces of $\ell_1(\Delta_i^3)$ and $bv(\Delta_i^3)$. Finally, we characterized the matrix classes $(\mu(\Delta_i^3)\colon \lambda)$ and $(\lambda\colon \mu(\Delta_i^3))$, where $\mu = \{\ell_1, bv\}$ and $\lambda = \{c, c_0, \ell_1, \ell_\infty, bs, cs, bv\}$, and we conclude the paper with some important results.

As a natural continuation of this paper, the fine spectrum and its subdivisions of the operator Δ_i^3 over the sequence spaces ℓ_1 and bv can be calculated. Moreover, the matrix domain of Δ_i^3 on the generalized Hahn sequence spaces h_d which was defined and studied by Malkowsky et al. [27] can also be calculated.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The author declares no conflicts of interest.

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