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DISCRETE LEAST SQUARE METHOD FOR SOLVING DIFFERENTIAL EQUATIONS

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Abstract

This paper investigates the least squares approach for finding approximate solutions to differential equations using discrete method. Our goal is to develop efficient numerical method (discrete method) for solving ordinary differential equations (ODEs). The L_2 norm along with the discrete least squares method (DLSM) has been used to obtain the least approximation error and numerical approximate solution, respectively. Some examples are given to support the explicit results.

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1. Introduction

In differential equation, the least squares method, also known as least squares approximation, is a method for estimating the true value of a quantity by considering errors in observations or measurements. The criterion for least squares is a formula used to calculate the precision of a straight line in the representation of the data used to produce it. That is, the formula decides the best-fit rows. The action of the dependent variables is predicted using this mathematical model. The method is also called the *regression line* of least squares. The criterion of least squares is estimated by minimizing the number of squares that a mathematical function produces.

The DLSM is an important issue in solving ODEs, which plays a great role in mathematical physics. The mixed discrete least squares meshless method for planar elasticity problems using regular and irregular nodal distributions was studied in [1, 2]. The discrete least squares meshless method with sampling points for the solution of elliptic partial differential equations was introduced in [3]. The reference [4] found the approximate solutions of first and second-order differential equations using continuous least squares method (CLSM). For comparison, some numerical methods for solving ordinary differential equations (ODEs), fractional differential equations (FDEs) and partial differential equations (PDEs) are studied in [5, 7]. The references [6-11] introduce numerical approximation approach that involves curves and surfaces which play a vital role in numerical analysis. As an application, references [12-16] made use of commutativity to study the relation and the sensitivity between systems. The idea can be extended to investigate the commutativity and sensitivity using the DLSM approach.

The aim of this paper is to promote numerical technique (discrete method) for ODEs. The L_2 norm along with the DLSM has been used to obtain the least approximation error and numerical approximate solution, respectively. The paper is scheduled as: Section 2 provides basic definitions and mathematical formulas. Section 3 introduces the CLSM and DLSM. Results and discussion form the content of Section 4. Finally, conclusion is given in Section 5.

2. Mathematical Preludes

CLSM is used to solve complex problems involving ODEs, FDEs and PDEs. In this work, the DLSM for solving ODEs is considered as:

$$L(y) = f(x) \text{ for } x \in \text{domain } \Omega,$$

$$W(y) = g(x) \text{ for } x \in \text{domain } \delta\Omega,$$

where L stands for differential operator and Ω indicates the domain in R^1 or R^2 or R^3, while W refers to the boundary operator. The approximate solution of ODEs can be written as

$$\tilde{y} = \sum_{i=1}^{n} q_i C^i(X), \tag{2.1}$$

where $C^i(X)$ and q_i represent the weighted basis function and the coefficients (weights), respectively, the q_i is realized using the DLSM. Let the residual $R_L(X)$ and $R_W(X)$ be defined as

$$R_L(x, \tilde{y}) = L(\tilde{y}) - f(x) \text{ for } x \in \text{domain } \Omega, \tag{2.2}$$

$$R_W(x, \tilde{y}) = w(\tilde{y}) - g(x) \text{ for } x \in \text{boundary } \delta\Omega. \tag{2.3}$$

Substitution of y_{exact} into equation (2.2) and equation (2.3) leads to $R_L(x, y_{exact}) = 0$ and $R_W(x, y_{exact}) = 0$.

3. Least Square Method

In this section, we introduce the CLSM and DLSM for solving ODEs.

3.1. Continuous least square method

The CLSM is a numerical approximation method that can be used to solve ODEs, the q_i's from equation (2.1) are obtained using the CLSM. This can be done by considering the L_2 norm:

$$E = \int_{\Omega} R_L^2(x, \tilde{y})dx + \int_{\alpha\Omega} R_W^2(x, \tilde{y})dx. \tag{3.1}$$

Minimizing the error function

We obtain the best numerical approximate solution along with minimal error E.

The first derivative of equation (3.1) with respect to q_i equated to zero yields

$$\frac{\partial E}{\partial q_i} = 0, \text{ for } i = 1, ..., N,$$

which, in turn, yields

$$\int_{\Omega} R_L(x, \tilde{y})\frac{\partial R_L}{\partial q_i} dx + \int_{\alpha\Omega} R_W(x, \tilde{y})\frac{\partial R_L}{\partial q_i} dx = 0, \quad i = 1, ..., N. \tag{3.2}$$

Equations in (3.2) are algebraic equations which can be written in the form of

$$Ma = b. \tag{3.3}$$

Note that M is an $n \times n$ matrix, $a = [q_1, q_2, q_3, ..., q_n]^T$, and b is some column vector.

3.2. Discrete least square method

The DLSM and the squared residuals are considered at finite points $x_i, 1 \le i \le k,$ in domain, and $x_i, k + 1 \le i \le m.$ Let

$$E = \sum_{i=1}^{k} R_L^2(x_i, \tilde{y}) + \sum_{i=k+1}^{m} R_L^2(x_i, \tilde{y}). \tag{3.4}$$

Using $\varphi^i(x),\ 1 \le i \le N,$ the solution of ODEs that present basic functions of an approximate solution of DLSM is expressed as:

$$r = \begin{pmatrix} R_L(a, x_1) \\ \vdots \\ R_L(a, x_k) \\ R_W(a, x_{k+1}) \\ \vdots \\ R_W(a, x_m) \end{pmatrix} = \begin{pmatrix} L\tilde{y}(a, x_1) - f(x_1) \\ \vdots \\ L\tilde{y}(a, x_k) - f(x_k) \\ L\tilde{y}(a, x_{k+1}) - f(x_{k+1}) \\ \vdots \\ L\tilde{y}(a, x_m) - f(x_m) \end{pmatrix}.$$

The solution of DLSM, which minimizes $E = r^T r,$ is obtained by

$$\frac{\partial E}{\partial q_i} = 0, \text{ for } i = 1, ..., N. \tag{3.5}$$

4. Results and Discussion

The explicit results obtained using DLSM in previous section are applied to show the effectiveness of our method.

4.1. Example 1

Application of the DLSM to solve 1st order ODE:

$$\left(\frac{1}{1+\sqrt{3}}\right)\frac{dy}{dx} + xy(x) = 0, \quad y(0) = 1, \tag{4.1}$$

where $0 \le x \le 1.$

Let

$$L(x, y) = \left(\frac{1}{1+\sqrt{3}}\right)\frac{dy}{dx} + xy. \tag{4.2}$$

Step 1. Apply the polynomial

$$\tilde{y} = \sum_{i=1}^{N} q_i x^i + y_0. \tag{4.3}$$

Step 2. We set $y_0 = 1$ in equation (4.3) to satisfy the boundary condition.

Step 3. The residual is given by

$$R(x) = \left(\frac{1}{1+\sqrt{3}}\right)\frac{d\tilde{y}}{dx} + x\tilde{y}. \tag{4.4}$$

By replacing $\tilde{y}(x)$ from equation (4.3) into equation (4.4), we get

$$R(x) = \left(\frac{1}{1+\sqrt{3}}\right)\frac{d\left(\sum_{i=1}^{N} q_i x_i + 1\right)}{dx} + x\left(\sum_{i=1}^{N} q_i x_i + 1\right). \tag{4.5}$$

Step 4. The least error is obtained by considering

$$E = \sum_{i=1}^{k} R_L^2(x_i, \tilde{y}) + \sum_{i=k+1}^{m} R_L^2(x_i, \tilde{y}). \tag{4.6}$$

Step 5. The discrete least square solution is obtained by solving equation (4.8):

$$\frac{\partial E}{\partial q_i} = 0, \text{ for } i = 1, \ldots, N, \tag{4.7}$$

$$\sum_{i=1}^{k} R_L(x_i, \tilde{y})\frac{\partial R_L(x_i, \tilde{y})}{\partial q_i} + \sum_{i=k+1}^{m} R_L(x_i, \tilde{y})\frac{\partial R_L(x_i, \tilde{y})}{\partial q_i} = 0, \; i = 1, \ldots, N. \tag{4.8}$$

The solutions obtained from (4.8) form a linear system. By choosing three discrete points $x_1 = 0$, $x_2 = 0.5$, $x_3 = 1$ and considering equation (4.5) with equation (4.8) for $N = 3$, with the help of MATLAB program, we obtain the matrix:

$$D = \begin{pmatrix} 4.7589 & 5.3370 & 6.1472 \\ 5.3372 & 6.4822 & 7.5989 \\ 6.1472 & 7.5989 & 9.0310 \end{pmatrix}, \quad b = \begin{pmatrix} 3.3481 \\ 3.9551 \\ 4.5331 \end{pmatrix}, \quad a = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}. \tag{4.9}$$

Solving equation (4.9) helps to obtain q_i's:

$$q_1 = 1.9072, \quad q_2 = 1.5947, \quad q_3 = 0.8399.$$

The approximate solution is given as

$$\tilde{y} = 0.8399x^3 - 1.5947x^2 - 1.9072x + 1. \tag{4.10}$$

The exact solution is given by

$$y_{exact} = e^{-\frac{1}{2}(1+\sqrt{3})x^2}. \tag{4.11}$$

The following figures depict the exact solution, DLSM and the error for $N = 3$.

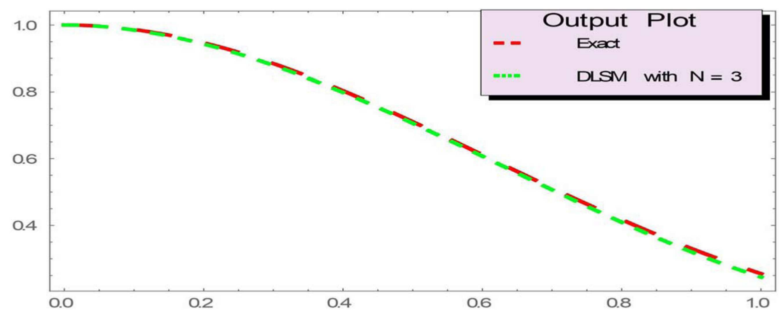

Figure 4.1. Results of equation (4.1) with $N = 3$.

The error is defined as

$$error = abs[y_{exact} - \tilde{y}]. \tag{4.12}$$

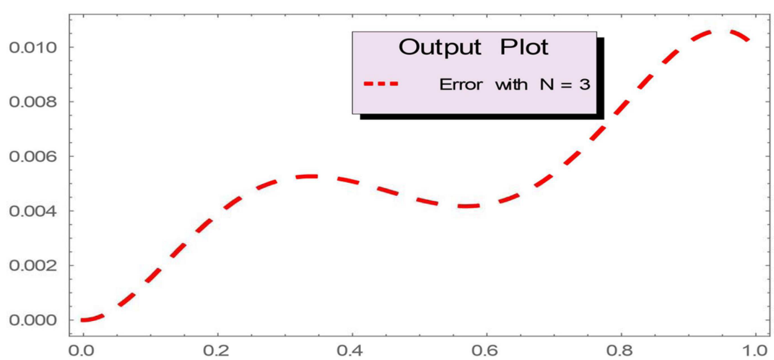

Figure 4.2. Error plot of equation (4.1) with $N = 3$.

For $N = 6$, we obtain the following matrix by choosing six discrete points $x_1 = 0$, $x_2 = 0.2$, $x_3 = 0.4$, $x_4 = 0.6$, $x_5 = 0.8$, $x_6 = 1$:

$$D = \begin{pmatrix} 8.0 & 8.4 & 9.0 & 9.6 & 10.2 & 10.9 \\ 8.4 & 9.6 & 10.6 & 11.5 & 12.4 & 13.4 \\ 9.0 & 10.6 & 12.0 & 13.2 & 14.4 & 15.7 \\ 9.6 & 11.5 & 13.2 & 14.8 & 16.4 & 17.9 \\ 10.2 & 12.4 & 14.4 & 16.4 & 18.2 & 20.1 \\ 10.9 & 13.4 & 15.7 & 18.0 & 20.1 & 22.2 \end{pmatrix}, \quad b = \begin{pmatrix} 5.8 \\ 6.4 \\ 6.8 \\ 7.2 \\ 7.7 \\ 8.1 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix}. \tag{4.13}$$

The approximate solution is

$$\tilde{y} = -0.7268x^5 + 1.50769x^4 - 0.1831x^3 - 1.3426x^2$$

$$+ 5.3532 \times 10^{-14} x + 1. \tag{4.14}$$

The following figures depict the exact solution, DLSM and the error for $N = 6$.

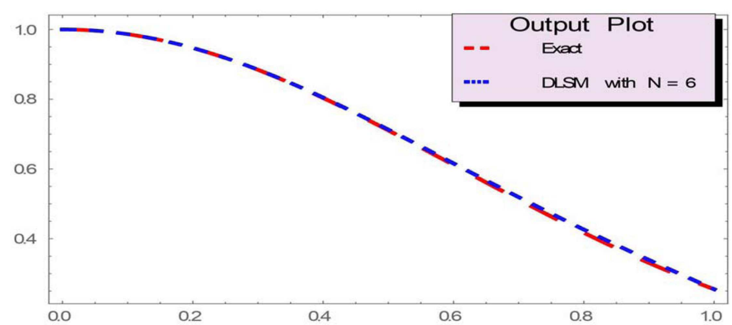

Figure 4.3. Results of equation (4.1) with $N = 6$.

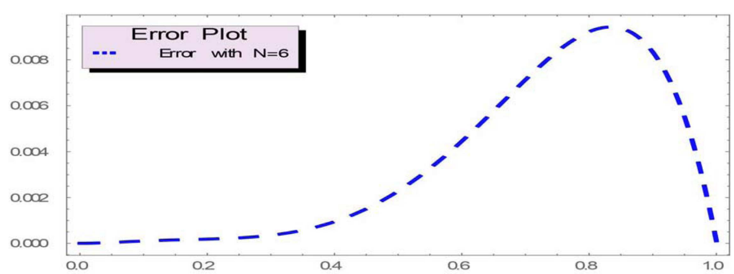

Figure 4.4. Error plot of equation (4.1) with $N = 6$.

The following figures depict the exact solution, DLSM and the error plot for $N = 3$ and $N = 6$.

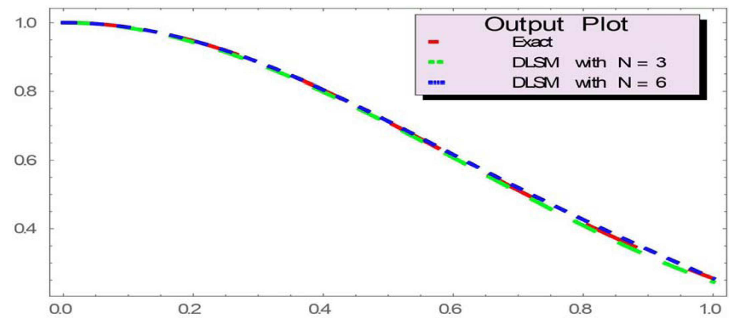

Figure 4.5. Results of equation (4.1) with $N = 3$ and $N = 6$.

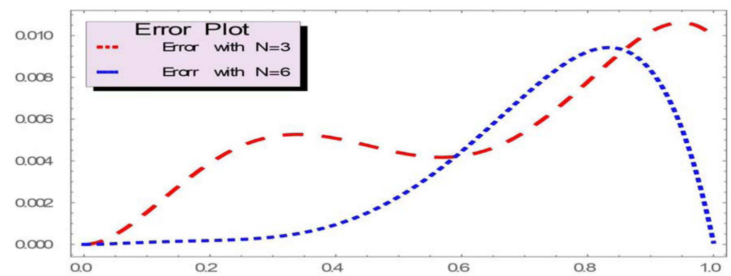

Figure 4.6. Error plots of equation (4.1) with $N = 3$ and $N = 6$.

Table 4.1. Data of errors analysis with $N = 3$ and $N = 6$ for 1st order ODE

x	y exact	y - DLSM with $N = 3$	y - DLSM with $N = 6$	Errors with $N = 3$	Errors with $N = 6$
0	1	1	1	0	0
0.1	0.986433	0.984892	0.972952	0.00154029	0.0134802
0.2	0.946825	0.942929	0.931079	0.00389601	0.0157463
0.3	0.884315	0.879149	0.870665	0.00516528	0.0136498
0.4	0.803672	0.798593	0.79209	0.00507886	0.0115819
0.5	0.710699	0.7063	0.699549	0.00439934	0.0111498
0.6	0.611544	0.607308	0.600776	0.00423521	0.0107671
0.7	0.51204	0.506659	0.506769	0.00538129	0.00527143
0.8	0.417172	0.40939	0.431508	0.0077818	0.0143362
0.9	0.330721	0.320543	0.391687	0.0101782	0.0609662
1.	0.255119	0.245155	0.406429	0.0099638	0.15131

4.2. Example 2

Application of DLSM to solve 2nd order ODE:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + (e^x + 0.2)\,y = 0, \quad y(0) = 1, \quad y^I(0) = 1, \tag{4.15}$$

where $0 \le x \le 1$.

Let

$$L(x,\ y) = \frac{d^2y}{dx^2} + \frac{dy}{dx} + (e^x + 0.2)\,y. \tag{4.16}$$

Step 1. Apply the polynomial

$$\tilde{y} = \sum_{i=1}^{N} q_i x^i + y_0. \tag{4.17}$$

Step 2. We set $y_0 = 11$ and $q_1 = 1$ in equation (4.17) to satisfy the boundary condition.

Step 3. The residual is given by

$$R(x) = \frac{d^2\tilde{y}}{dx^2} + \frac{d\tilde{y}}{dx} + (e^x + 0.2)\,\tilde{y}. \tag{4.18}$$

By replacing $\tilde{y}(x)$ from equation (4.17) into equation (4.18), we obtain:

$$R(x) = \frac{d^2\left(\sum_{i=1}^{N} q_i x_i + 1\right)}{dx^2} + \frac{d\left(\sum_{i=1}^{N} q_i x_i + 1\right)}{dx} + (e^x + 0.2)\left(\sum_{i=1}^{N} q_i x_i + 1\right). \tag{4.19}$$

Step 4. The least error is obtained by considering

$$E = \sum_{i=1}^{k} R_L^2(x_i,\ \tilde{y}) + \sum_{i=k+1}^{m} R_L^2(x_i,\ \tilde{y}). \tag{4.20}$$

Step 5. The discrete least square solution is obtained by solving equation (4.20):

$$\frac{\partial E}{\partial q_i} = 0, \text{ for } i = 1, \dots, N, \tag{4.21}$$

$$\sum_{i=1}^{k} R_L(x_i, \tilde{y}) \frac{\partial R_L(x_i, \tilde{y})}{\partial q_i} + \sum_{i=k+1}^{m} R_L(x_i, \tilde{y}) \frac{\partial R_L(x_i, \tilde{y})}{\partial q_i} = 0, \quad i = 1, \dots, N. \tag{4.22}$$

The solutions obtained from (4.22) form a linear system. By choosing three discrete points $x_1 = 0$, $x_2 = 0.5$, $x_3 = 1$ and considering equation (4.19) with equation (4.22) for $N = 3$, with the help of MATLAB program, we obtain the matrix:

$$D = \begin{pmatrix} 127.699 & 192.4745 \\ 192.475 & 315.789 \end{pmatrix}, \quad b = \begin{pmatrix} 129.521 \\ 193.002 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \end{pmatrix}. \tag{4.23}$$

The approximate solution is

$$\tilde{y} = 0.0863998x^3 - 1.1445x^2 + x + 1. \tag{4.24}$$

The exact solution is given by

$$y_{exact} = -3.61363 \times 2.71828^{-0.5x}(\text{BesselJ}[-0.447214,\ 2\sqrt{2.71828^x}]$$

$$- 0.185917\,\text{BesselJ}[-0.447214,\ 2\sqrt{2.71828^x}]). \tag{4.25}$$

The approximate exact solution and the error are depicted in Figure 4.7 and Figure 4.8 for $N = 3$.

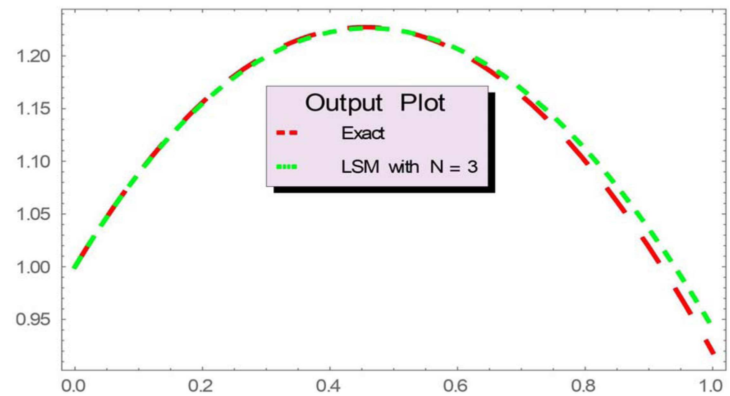

Figure 4.7. Result of equation (4.15) with $N = 3$.

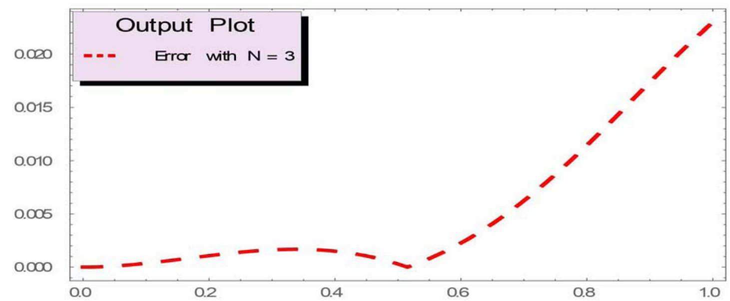

Figure 4.8. Error plots of equation (4.15) with $N = 3$.

For $N = 6$, we obtain the following matrix by choosing six discrete points $x_1 = 0$, $x_2 = 0.2$, $x_3 = 0.4$, $x_4 = 0.6$, $x_5 = 0.8$ and $x_6 = 1$:

$$D = \begin{pmatrix} 218.6 & 312.0 & 431.2 & 571.0 & 732.3 \\ 312.0 & 484.6 & 692.0 & 935.2 & 1215 \\ 431.2 & 692.0 & 1015 & 1399 & 1844 \\ 570.9 & 935.2 & 1399 & 1957 & 2610 \\ 732.3 & 1215 & 1844 & 2610 & 3513 \end{pmatrix}, \quad b = \begin{pmatrix} 226.0 \\ 319.1 \\ 437.6 \\ 576.1 \\ 735.5 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix}. \tag{4.26}$$

The approximate solution is

$$\begin{aligned} \tilde{y} &= 0.00932378x^6 + 0.0257951x^5 - 0.0166895x^4 \\ &\quad + 0.00053x^3 - 1.10001x^2 + x + 1. \end{aligned} \tag{4.27}$$

The approximate exact solution and the error are depicted in Figure 4.9 and Figure 4.10 for $N = 6$.

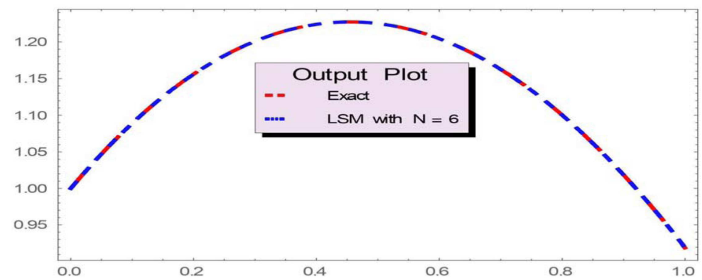

Figure 4.9. Results of equation (4.15) with $N = 6$.

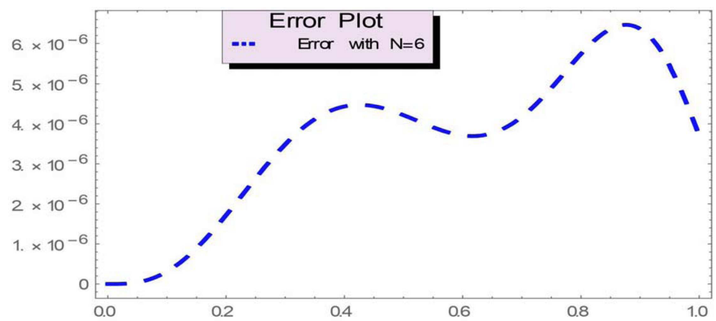

Figure 4.10. Error plots of equation (4.15) with $N = 6$.

The comparison between the exact DLSM and error is depicted in Figure 4.11 and Figure 4.12 for $N = 3$ and $N = 6$.

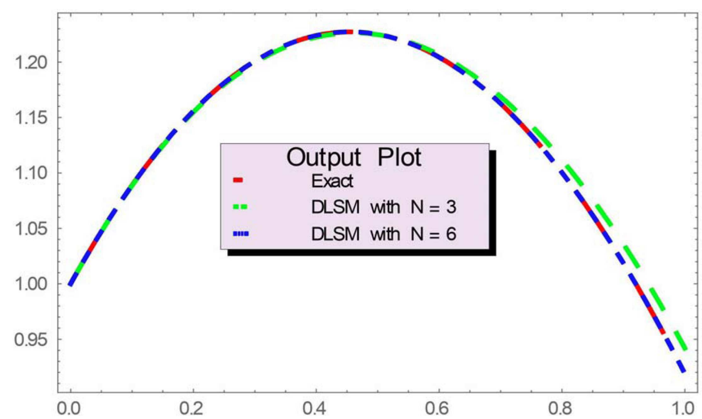

Figure 4.11. Results of equation (4.15) with $N = 3$ and $N = 6$.

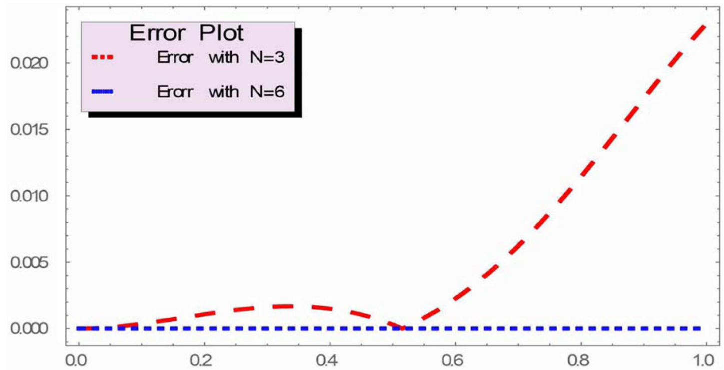

Figure 4.12. Error plots of equation (4.15) with $N = 3$ and $N = 6$.

Table 4.2. Data of error analysis with $N = 3$ and $N = 6$ for 2nd order ODE

x	y exact	y - DLSM. with $N = 3$	y - DLSM. with $N = 6$	Errors with $N = 3$	Errors with $N = 6$
0	1	1	1	0	0
0.1	1.089	1.08864	1.089	0.000357	3.1102×10^{-7}
0.2	1.15598	1.15491	1.15599	0.001073	1.7184×10^{-6}
0.3	1.20094	1.19933	1.20095	0.001616	3.4918×10^{-6}
0.4	1.2239	1.22241	1.22391	0.00149	4.43531×10^{-6}
0.5	1.22497	1.22468	1.22497	0.000293	4.21863×10^{-6}
0.6	1.20439	1.20664	1.20439	0.002258	3.7113×10^{-6}
07	1.1626	1.16883	1.1626	0.006232	4.17652×10^{-6}
0.8	1.10032	1.11176	1.10033	0.01144	5.74019×10^{-6}
0.9	1.01861	1.03594	1.01862	0.01733	6.36734×10^{-6}
1.0	0.91895	0.941904	0.918953	0.02296	3.70363×10^{-6}

5. Conclusion

This paper investigates the discrete least square method (DLSM) for solving differential equations. The DLSM was introduced along with the L_2 norm in order to obtain better numerical approximation results with least error. We consider first order and second order ODEs with $N = 3, 6$ to verify the effectiveness of the DLSM. Our method to obtain results is seen to outperform the existing methods with small error.

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