Advances in Differential Equations and Control Processes
© 2022 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324322037
Volume 29, 2022, Pages 127-138 **P-ISSN: 0974-3243**

OPTICAL SOLITON SOLUTIONS FOR THE NONLINEAR THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION

Salisu Ibrahim

Mathematics Education
Tishk International University-Erbil
Kurdistan Region, Iraq

Abstract

In this paper, the Riccati-Bernoulli (RB) sub-ODE method is used to find the solitary wave solutions for a third-order nonlinear partial differential equation (NLPDE). The traveling wave transformation along with RB sub-ODE equation is used to convert the third-order NLPDE to the set of algebraic equations. Solving the set of algebraic equations generates the analytical solution of the third-order NLPDE. The RB sub-ODE method is a powerful and simple mathematical tool for solving complex NLPDE. The solitary wave solutions obtained play a vital role in mathematical physics.

Received: November 4, 2022; Revised: November 30, 2022; Accepted: December 15, 2022
2020 Mathematics Subject Classification: 35C07, 35C08, 35D35.
Keywords and phrases: third-order nonlinear equation, optical solitons, traveling wave solutions, Riccati-Bernoulli sub-ODE method.

How to cite this article: Salisu Ibrahim, Optical soliton solutions for the nonlinear third-order partial differential equation, Advances in Differential Equations and Control Processes 29 (2022), 127-138. http://dx.doi.org/10.17654/0974324322037

Published Online: December 22, 2022

1. Introduction

The process of finding the analytical solution to NLPDEs by using different computational techniques, theoretical methods, and numerical methods has been the major challenge of mathematicians and physicist [1-3]. This is because of the continuous application of NLPDEs in different areas of study such as applied astronomy, engineering, science, physics, chemistry, and biological [4-6]. Certain techniques, theories, models, and methods to find exact solutions of different NLPDEs are available in [7-9, 12]. The NLPDEs describe real life phenomena that deal with physical systems and their solutions [10-13]. These are constantly used in chaos theory for dynamical systems, quantum theory, fluid dynamics, continuum mechanics, nonlinear optics, and other related areas [14, 15]. Commutativity of NLPDEs is an open problem [16-20].

The goal of this paper is to investigate the exact solution of traveling wave solution of the third-order NLPDEs using RB-sub ODE method.

Consider the third-order $(1+1)$-dimensional equations as

$$\vartheta_t = -\alpha\vartheta\vartheta_x - \beta\vartheta_{xxx} - \gamma(\vartheta\vartheta_{xx})_x - d\vartheta_x\vartheta_{xx}, \quad (1)$$

where α, β, γ and d are nonzero real parameters. The RB-sub ODE method was introduced to deal with the problems of exact solution of complex NLPDEs, because of its simplicity and ease for computation. Many authors make use of this technique on different NLPDEs [21-23].

Regarding this work, we analogously use the RB-sub ODE method to investigate the traveling wave solution of the third-order NLPDEs. We study the analytical solution of this novel third-order NLPDEs using this method.

The paper is scheduled as: Section 2 introduces the method. The application and figures are given in Section 3. Lastly, Section 4 presents the conclusion.

2. Description of RB Sub-ODE Method

In this section, we offer the RB sub-equation method. Suppose we have a NLPDE as

$$P(\vartheta, \vartheta_t, \vartheta_x, \vartheta_{tt}, \vartheta_{xx}, \vartheta_{tx}, \ldots) = 0, \tag{2}$$

where P is a polynomial. The RB sub-equation method is categorized into three steps.

Step 1. We consider the following traveling wave transformation:

$$\vartheta(\xi) = \vartheta(x, t), \quad \xi = K(x \pm vt) \tag{3}$$

that leads to the following ODE:

$$P(\vartheta, \vartheta', \vartheta'', \ldots) = 0, \tag{4}$$

where $\vartheta'(\xi) = \frac{d\vartheta}{d\xi}$.

Step 2. Let equation (4) be the solution of the RB equation

$$\vartheta' = b\vartheta + a\vartheta^{2-m} + c\vartheta^{m}, \tag{5}$$

where a, b, c and m are arbitrary constants.

Differentiating equation (5) leads to

$$\vartheta'' = \vartheta^{-1-2m}(a\vartheta^2 + c\vartheta^{2m} + b\vartheta^{1+m}) \times(-a(-2+m)U^2 + cm\vartheta^{2m} + b\vartheta^{1+m}), \tag{6}$$

$$\vartheta''' = \vartheta^{-2(1+m)}(bu + a\vartheta^{2-m} + c\vartheta^{m})(a^2(-2+m)(-3+2m)\vartheta^4 + c^2 m(-1+2m)\vartheta^{4m} + ab(-3+m)(-2+m)\vartheta^{3+m} + (b^2+2ac)\vartheta^{2+2m} + bcm(1+m)\vartheta^{1+3m}), \tag{7}$$

and so on.

Observe that the solutions of equation (5) lead to

Case 1. As $m = 1,$ the results of equation (5) become

$$\vartheta(\xi) = Je^{(b+a+c)\xi}. \tag{8}$$

Case 2. As $m \neq 1,$ $b = 0$ and $c = 0,$ the results of equation (5) become

$$\vartheta(\xi) = (a(m-1)(\xi + J))^{\frac{1}{m-1}}. \tag{9}$$

Case 3. As $m \neq 1,$ $b \neq 0$ and $c = 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(Je^{(b(m-1)\xi)} - \frac{a}{b}\right)^{\frac{1}{m-1}}. \tag{10}$$

Case 4. As $m \neq 1,$ $a \neq 0$ and $b^2 - 4ac < 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}\tan\left[\frac{(1-m)\sqrt{4ac - b^2}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}} \tag{11}$$

and

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}\cot\left[\frac{(1-m)\sqrt{4ac - b^2}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}}. \tag{12}$$

Case 5. As $m \neq 1,$ $a \neq 0$ and $b^2 - 4ac > 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\tanh\left[\frac{(1-m)\sqrt{b^2 - 4ac}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}} \tag{13}$$

and

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\coth\left[\frac{(1-m)\sqrt{b^2-4ac}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}}. \quad (14)$$

Case 6. As $m \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, the results of equation (5) become

$$\vartheta(\xi) = \left(\frac{1}{a(m-1)(\xi + J)} - \frac{b}{2a}\right)^{\frac{1}{1-m}}, \quad (15)$$

where J is a constant.

Step 3. Plugging the derivatives of ϑ into equation (4) gives the equation in terms of ϑ. Collecting terms that belong together and solving for the unknown constants provides the solution of equation (2), see [22].

2.1. Bäcklund transformation

Suppose that $\vartheta_n(\xi)$ and $\vartheta_{n-1}(\xi)$ are the solutions of equation (2). Then

$$\frac{d\vartheta_n(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}(\xi)\xi}\frac{d\vartheta_{n-1}(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}\xi}(a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^{m}), \quad (16)$$

namely,

$$\frac{d\vartheta_n(\xi)}{a\vartheta_n^{2-m} + b\vartheta_n + c\vartheta_n^{m}} = \frac{d\vartheta_{n-1}(\xi)}{a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^{m}}. \quad (17)$$

Integrating equation (17) with respect to ξ leads to

$$\vartheta_n(\xi) = \left(\frac{-cA_1 + aA_2(\vartheta_{n-1}(\xi))^{1-m}}{bA_1 + aA_2 + aA_1(\vartheta_{n-1}(\xi))^{1-m}}\right)^{\frac{1}{1-m}}, \quad (18)$$

where A_1 and A_2 are arbitrary constants. With equation (18), we can obtain the solution of equation (2), and the process is called a *Bäcklund transformation*.

3. Applications

To get the solution of third-order NLPDE given in equation (1), we consider the traveling wave transformation

$$\vartheta(x, t) = \vartheta(\xi), \quad \xi = K(x + vt), \tag{19}$$

and use it into equation (1). We get the following equation:

$$Kv\vartheta' = -K\alpha\vartheta\vartheta' - dK^3\vartheta'\vartheta'' - K^3\beta\vartheta^{(3)} - \gamma(K^3\vartheta'\vartheta'' + K^3\vartheta\vartheta^{(3)}). \tag{20}$$

Plugging equations (5)-(7) and its derivative into (20), setting $m = 0$ and collecting all the coefficients of $U^i(\xi)$ (for $i = 0, 1, 2, 3, 4, 5$), and also equating each collection to zero, we have the following:

$$\vartheta^0(\xi) : ck(v + b^2k^2\beta + 2ack^2\beta + bck^2(d + \gamma)) = 0,$$

$$\vartheta^1(\xi) : k\,(b^3k^2\beta + b(v + 8ack^2\beta) + b^2ck^2(2d + 3\gamma)$$
$$+ c(\alpha + 2ack^2(d + 2\gamma))) = 0,$$

$$\vartheta^2(\xi) : k\,(7ab^2k^2\beta + a\,(v + 8ack^2\beta) + b^3k^2(d + 2\gamma)$$
$$+ b(\alpha + 2ack^2(3d + 7\gamma))) = 0,$$

$$\vartheta^3(\xi) : ak\,(\alpha + 12abk^2\beta + 4ack^2(d + 3\gamma) + b^2k^2(4d + 11\gamma)) = 0,$$

$$\vartheta^4(\xi) : a^2k^3(5bd + 6\alpha\beta + 17b\gamma) = 0,$$

$$\vartheta^5(\xi) : 2a^3k^3(d + 4\gamma) = 0. \tag{21}$$

Solving the system of algebraic equations of equation (21) leads to

$$a = \frac{ck^2\gamma^2 + \sqrt{-k^2\alpha\beta^2\gamma + c^2k^4\gamma^4}}{2k^2\beta^2},$$

$$b = \frac{2a\beta}{\gamma},$$

$$v = -(b^2 - 4ac)k^2\beta,$$

$$K = \frac{1}{144}(3\alpha + 2\beta + 2\gamma)^2,$$

$$d = -4\gamma. \tag{22}$$

Considering the solutions of equation (22) with equations (8)-(15) and (19), we obtain the solutions of equation (1).

The periodic solution can be given by

$$\vartheta_1^{\pm}(x, t) = -\frac{\beta}{\gamma} - \frac{\beta}{\gamma^2}\tanh\left[\frac{1}{2k}\sqrt{\frac{-\alpha}{\gamma}}\left(J + k\left(x + \frac{t\beta\alpha}{\gamma}\right)\right)\right], \tag{23}$$

$$\vartheta_2^{\pm}(x, t) = -\frac{\beta}{\gamma} + \frac{ik\beta\sqrt{\frac{\alpha}{k^2\gamma}}\cot\left[\frac{1}{2}\left(J + k\left(x + \frac{t\alpha\beta}{\gamma}\right)\right)\sqrt{\frac{\alpha}{k^2\gamma}}\right]}{\sqrt{\alpha}\sqrt{\gamma}}. \tag{24}$$

The dark optical soliton:

$$\vartheta_3^{\pm}(x, t) = -\frac{\beta}{\gamma} + \frac{ik\beta\sqrt{-\frac{\alpha}{k^2\gamma}}\coth\left[\frac{1}{2}\left(J + k\left(x + \frac{t\alpha\beta}{\gamma}\right)\right)\sqrt{-\frac{\alpha}{k^2\gamma}}\right]}{\sqrt{\alpha}\sqrt{\gamma}}, \tag{25}$$

and the singular soliton:

$$\vartheta_4^{\pm}(x, t) = -\frac{\beta}{\gamma} + \frac{ik\beta\sqrt{-\frac{\alpha}{k^2\gamma}}\tanh\left[\frac{1}{2}\left(J + k\left(x + \frac{t\alpha\beta}{\gamma}\right)\right)\sqrt{-\frac{\alpha}{k^2\gamma}}\right]}{\sqrt{\alpha}\sqrt{\gamma}}, \tag{26}$$

$$\vartheta_5^{\pm}(x, t) = \frac{1}{e^{-\frac{i\sqrt{\alpha}\left(x+\frac{t\alpha\beta}{\gamma}\right)}{\sqrt{\gamma}}}J - \frac{\gamma}{2\beta}}, \tag{27}$$

$$\vartheta_6^{\pm}(x,\, t) = \frac{e^{\frac{k\sqrt{\alpha}\left(x - \frac{k^2 t\alpha\beta}{-dk^2 - 2k^2\gamma}\right)}{\sqrt{-dk^2 - 2k^2\gamma}}}}{J}. \tag{28}$$

Figure 1 presents the periodic singular wave solution, that is $\vartheta_1(x,\, t)$ of equation (23). We analogously consider the following parameters:

$$c = 6; \quad \alpha = 8; \quad \gamma = 10; \quad \beta = -0.5; \quad \mu = 7; \quad J = -0.4.$$

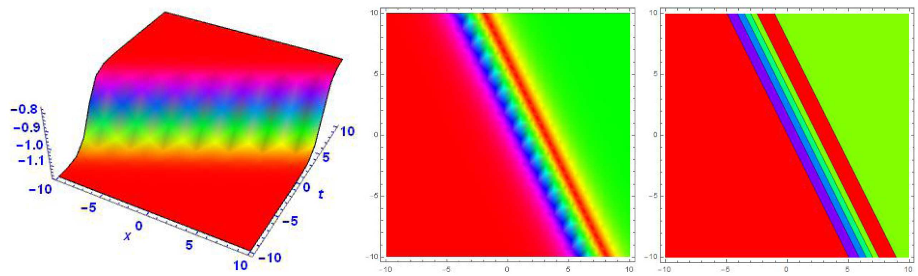

Figure 1. Plot of 3D, density and contour of (23).

Figure 2 presents the periodic singular wave solution, that is $\vartheta_2(x,\, t)$ of equation (22). We analogously consider the following parameters:

$$c = -6; \quad \alpha = -5; \quad \gamma = -10; \quad \beta = 0.5; \quad \mu = 0.25; \quad J = 1.5.$$

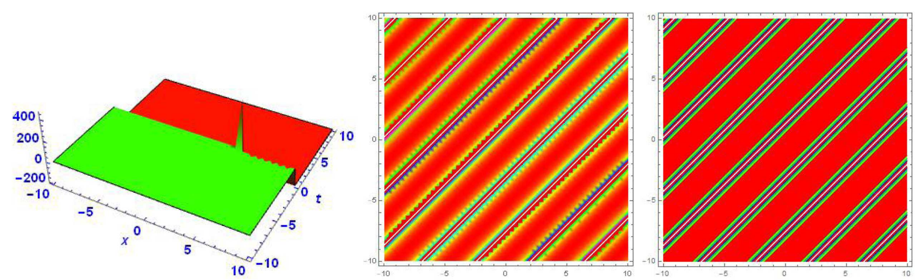

Figure 2. Plot of 3D, density and contour of (24).

Figure 3 presents the dark soliton solution, that is $\vartheta_3(x,\, t)$ of equation (25). We analogously consider the following parameters:

$$c = -6; \quad \alpha = -10; \quad \gamma = -10; \quad \beta = 2; \quad \mu = 0.25; \quad J = 1.5$$

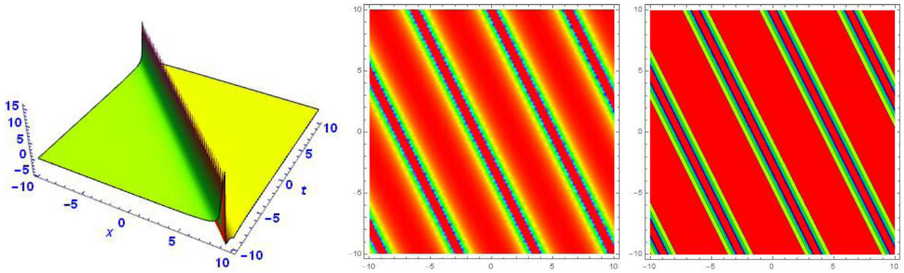

Figure 3. Plot of 3D, density and contour of (25).

Figure 4 presents the singular soliton solution, that is $\vartheta_4(x, t)$ of equation (26). We analogously consider the following parameters:

$$c = 6; \quad \alpha = -10; \quad \gamma = -10; \quad \beta = 2; \quad \mu = 0.25; \quad J = 1.5.$$

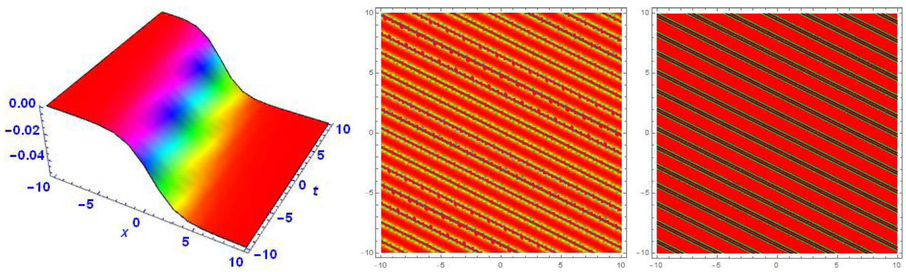

Figure 4. Plot of 3D, density and contour of (26).

Figure 5 presents the periodic solution, that is $\vartheta_5(x, t)$ of equation (27). We analogously consider the following parameters:

$$c = -6; \quad \alpha = -5; \quad \gamma = -10; \quad \beta = -4; \quad \mu = 0.9; \quad J = 1.$$

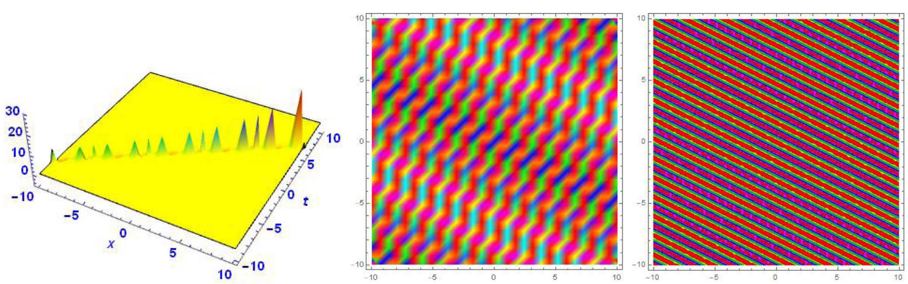

Figure 5. Plot of 3D, density and contour of (27).

Figure 6 presents the periodic solution, that is $\vartheta_6(x, t)$ of equation (28). We analogously consider the following parameters:

$$c = -6; \quad \alpha = -5; \quad \gamma = -10; \quad \beta = -4; \quad \mu = 0.9; \quad J = 1.$$

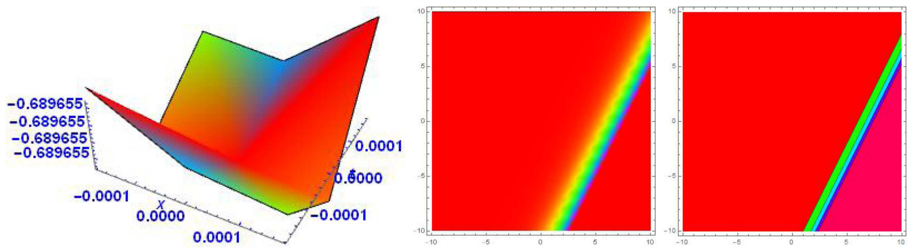

Figure 6. Plot of 3D, density and contour of (28).

4. Concluding Remarks

In this paper, we investigate the exact solution of the third-order NLPDE using RB sub-ODE method. The obtained solitary wave solution secured some singular solitons, periodic wave solutions and dark solitons to the third-order NLPDE. Moreover, the RB sub-ODE method is a simple mathematical tool that is used in mathematics to solve many complex NLPDEs. The traveling wave solutions obtained by this method have applications in mathematical physics. The results obtained are depicted in figures and validated with Mathematica software.

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions and constructive criticisms which improved the entire paper.

References

[1] G. B. Whitham, Linear and Nonlinear Waves, John Wiley and Sons, 2011.

[2] G. P. Agrawal, Nonlinear fiber optics, Nonlinear Science at the Dawn of the 21st Century, Springer, Berlin, Heidelberg, 2000.

[3] A. Hasegawa, Y. Kodama and A. Maruta, Recent progress in dispersion-managed soliton transmission technologies, Optical Fiber Technology 3(3) (1997), 197-213.

[4] F. Tchier, A. I. Aliyu, A. Yusuf and M. Inc, Dynamics of solitons to the ill-posed Boussinesq equation, The European Physical Journal Plus 132(3) (2017), 1-9.

[5] M. Mirzazadeh, M. F. Mahmood, F. B. Majid, A. Biswas and M. Belic, Optical solitons in birefringent fibers with Riccati equation method, Optoelectron. Adv. Mater.-Rapid Commun. 9 (2015), 1032-1036.

[6] S. Ibrahim, T. A. Sulaiman, A. Yusuf, A. S. Alshomrani and D. Baleanu, Families of optical soliton solutions for the nonlinear Hirota-Schrodinger equation, Opt. Quant. Electron 54(11) (2022), 1-15. https://doi.org/10.1007/s11082-022-04149-x.

[7] F. Tchier, A. Yusuf, A. I. Aliyu and M. Inc, Soliton solutions and conservation laws for lossy nonlinear transmission line equation, Superlattices and Microstructures 107 (2017), 320-336.

[8] T. A. Sulaiman, A. Yusuf, A. S. Alshomrani and D. Baleanu, Lump collision phenomena to a nonlinear physical model in coastal engineering, Mathematics 10(15) (2022), p. 2805.

[9] T. A. Sulaiman, U. Younas, M. Younis, J. Ahmad, S. U. Rehman, M. Bilal and A. Yusuf, Modulation instability analysis, optical solitons and other solutions to the $(2+1)$-dimensional hyperbolic nonlinear Schrodinger's equation, Computational Methods for Differential Equations 10(1) (2022), 179-190.

[10] J. J. Fang, D. S. Mou, H. C. Zhang and Y. Y. Wang, Discrete fractional soliton dynamics of the fractional Ablowitz-Ladik model, Optik 228 (2021), 166186.

[11] G. Tao, J. Sabi'u, S. Nestor, R. M. El-Shiekh, L. Akinyemi, E. Az-Zo'bi and G. Betchewe, Dynamics of a new class of solitary wave structures in telecommunications systems via a $(2+1)$-dimensional nonlinear transmission line, Modern Phys. Lett. B 36(19) (2022), 2150596.

[12] S. Ibrahim, Discrete least square method for solving differential equations, Advances and Applications in Discrete Mathematics 30 (2022), 87-102. http://dx.doi.org/10.17654/0974165822021.

[13] A. D. Khalaf, A. Zeb, Y. A. Sabawi, S. Djilali and X. Wang, Optimal rates for the parameter prediction of a Gaussian Vasicek process, The European Physical Journal Plus 136(8) (2021), 1-17.

[14] L. Akinyemi, U. Akpan, P. Veeresha, H. Rezazadeh and M. Inc, Computational techniques to study the dynamics of generalized unstable nonlinear Schrödinger equation, Journal of Ocean Engineering and Science (2022), 1-18. https://doi.org/10.1016/j.joes.2022.02.011.

[15] Y. A. Sabawi, A posteriori error analysis in finite element approximation for fully discrete semilinear parabolic problems, Finite Element Methods and their Applications, IntechOpen, 2020, pp. 1-19.

[16] S. Ibrahim and M. E. Koksal, Commutativity of sixth-order time-varying linear systems, Circuits, Systems, and Signal Processing 40(10) (2021), 4799-4832.

[17] S. Ibrahim and M. E. Koksal, Realization of a fourth-order linear time-varying differential system with nonzero initial conditions by cascaded two second-order commutative pairs, Circuits, Systems, and Signal Processing 40(6) (2021), 3107-3123.

[18] S. Ibrahim and A. Rababah, Decomposition of fourth-order Euler-type linear time-varying differential system into cascaded two second-order Euler commutative pairs, Complexity Volume 2022, Article ID 3690019, 9 pp. https://doi.org/10.1155/2022/3690019.

[19] S. Ibrahim, Commutativity of high-order linear time-varying systems, Advances in Differential Equations and Control Processes 27 (2022), 73-83. http://dx.doi.org/10.17654/0974324322013.

[20] S. Ibrahim, Commutativity associated with Euler second-order differential equation, Advances in Differential Equations and Control Processes 28 (2022), 29-36. http://dx.doi.org/10.17654/0974324322022.

[21] X. F. Yang, Z. C. Deng and Y. Wei, A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application, Advances in Difference Equations 2015(1) (2015), 1-17.

[22] D. Baleanu, M. Inc, A. I. Aliyu and A. Yusuf, Dark optical solitons and conservation laws to the resonance nonlinear Schrödinger's equation with Kerr law nonlinearity, Optik 147 (2017), 248-255.

[23] B. Karaman, New exact solutions of the time-fractional foam drainage equation via a Riccati-Bernoulli sub ode method, Online International Symposium on Applied Mathematics and Engineering (ISAME22), Istanbul-Turkey, 2022, 105 pp.