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SOLITARY WAVE SOLUTIONS FOR THE $(2+1)$ CBS EQUATION

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Abstract

The aim of this paper is to investigate the traveling wave solution of the Calogero-Bogoyavlenskii-Schiff (CBS) equation using the Riccati-Bernoulli (RB) sub-ODE method. The (RB) sub-ODE method is used to secure traveling wave solutions that are expressed explicitly and graphically in 3D. The RB sub-ODE technique is a powerful tool that is used to solve various nonlinear partial differential equations (NPDEs). The obtained soliton solutions have been demonstrated by relevant figures.

1. Introduction

The NPDEs have become the leading and the most important topics in mathematical physics that describes the nonlinear wave structure and

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behaviour which has lead to a great contribution in science and technology [1-4]. Because of the difficulty to find exact solution to NPDEs problems, many computational techniques and theoretical methods have been tackled by many researchers to come up with a great contribution, models, finding, theories, techniques, and methods to find solutions to NPDEs problems [5-7]. The significant and useful aspect of the solutions of NPDEs has unlimited contribution in various fields of study such as nonlinear optics, fluid mechanics, solid state physics, neural physics, mathematical biology, quantum mechanics, chaos, hydrodynamics, optical fibers and other numerous areas [8-10]. Moreover, the concept of NPDEs can be extended to the study of commutativity of NPDEs [11-15].

The aim of this paper is to investigate the exact traveling wave solution of the $(2+1)$-dimensional (CBS) equation using (RB) sub-ODE method.

Consider the $(2+1)$-dimensional (CBS) equation as

$$\vartheta_{xt} + 4\vartheta_x\vartheta_{xy} + 2\vartheta_{xx}\vartheta_y + \vartheta_{xxxy} = 0. \tag{1}$$

The RB-sub ODE method has been used to solve different NPDEs [16-18]. Exact solution and the applications of CBS equation are presented in [19].

In the present article, we consider the RB-sub ODE method to find the traveling wave solution of the $(2+1)$-dimensional (CBS) equation. The exact solution of this novel $(2+1)$-dimensional (CBS) equation using the RB-sub ODE method has not yet appeared in the literature. The paper is scheduled as: Section 2 introduces the method. The application and figures are given in Section 3. Lastly, Section 4 presents the conclusion.

2. Description of RB Sub-ODE Method

In this section, we propose the RB sub-equation method. Suppose we have NPDEs as

$$P(\vartheta, \vartheta_t, \vartheta_x, \vartheta_y, \vartheta_{tt}, \vartheta_{xt}, \vartheta_{xx}, \vartheta_{ty}, \vartheta_{xy}, \vartheta_{yy}, \ldots) = 0, \tag{2}$$

where P is a polynomial. The RB sub-equation method is described into three steps:

Step 1. We consider the following traveling wave transformation:

$$\vartheta(\xi) = \vartheta(x, y, t), \quad \xi = (x + y \pm vt), \tag{3}$$

that leads to the following ODE

$$P(\vartheta, \vartheta', \vartheta'', ...) = 0, \tag{4}$$

where $\vartheta'(\xi) = \frac{d\vartheta}{d\xi}$.

Step 2. Let equation (4) be the solution of the RB equation

$$\vartheta' = b\vartheta + a\vartheta^{2-m} + c\vartheta^{m}, \tag{5}$$

where a, b, c and m are arbitrary constants.

Differentiating equation (5) leads to

$$\vartheta'' = \vartheta^{-1-2m}(a\vartheta^2 + c\vartheta^{2m} + b\vartheta^{1+m})$$
$$\times(-a(-2+m)U^2 + cm\vartheta^{2m} + b\vartheta^{1+m}), \tag{6}$$

$$\vartheta''' = \vartheta^{-2(1+m)}(bu + a\vartheta^{2-m} + c\vartheta^{m})(a^2(-2+m)(-3+2m)\vartheta^4$$
$$+ c^2m(-1+2m)\vartheta^{4m} + ab(-3+m)(-2+m)\vartheta^{3+m}$$
$$+ (b^2 + 2ac)\vartheta^{2+2m} + bcm(1+m)\vartheta^{1+3m}), \tag{7}$$

and so on.

Observe that the solutions of equation (5) lead to

Case 1. As $m = 1$, the results of equation (5) become

$$\vartheta(\xi) = Je^{(b+a+c)\xi}. \tag{8}$$

Case 2. As $m \neq 1$, $b = 0$ and $c = 0$, the results of equation (5) become

$$\vartheta(\xi) = (a(m-1)(\xi + J))^{\frac{1}{m-1}}. \tag{9}$$

Case 3. As $m \neq 1,\ b \neq 0$ and $c = 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(Je^{(b(m-1)\xi)} - \frac{a}{b}\right)^{\frac{1}{m-1}}. \tag{10}$$

Case 4. As $m \neq 1,\ a \neq 0$ and $b^2 - 4ac < 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}\tan\left[\frac{(1-m)\sqrt{4ac - b^2}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}} \tag{11}$$

and

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}\cot\left[\frac{(1-m)\sqrt{4ac - b^2}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}}. \tag{12}$$

Case 5. As $m \neq 1,\ a \neq 0$ and $b^2 - 4ac > 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\tanh\left[\frac{(1-m)\sqrt{b^2 - 4ac}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}} \tag{13}$$

and

$$\vartheta(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\coth\left[\frac{(1-m)\sqrt{b^2 - 4ac}}{2}(\xi + J)\right]\right)^{\frac{1}{1-m}}. \tag{14}$$

Case 6. As $m \neq 1,\ a \neq 0$ and $b^2 - 4ac = 0,$ the results of equation (5) become

$$\vartheta(\xi) = \left(\frac{1}{a(m-1)(\xi + J)} - \frac{b}{2a}\right)^{\frac{1}{1-m}}, \tag{15}$$

where J is a constant.

Step 3. Plugging the derivatives of ϑ into equation (4) gives the equation in terms of ϑ. Collecting terms that belong together and solving for the unknowns constants leads to the solution of equation (2).

2.1. Bäcklund transformation

Suppose that $\vartheta_n(\xi)$ and $\vartheta_{n-1}(\xi)$ are the solutions of equation (2). Then

$$\frac{d\vartheta_n(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}(\xi)\xi}\frac{d\vartheta_{n-1}(\xi)}{d\xi} = \frac{d\vartheta_n(\xi)}{d\vartheta_{n-1}\xi}(a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^{m}), \quad (16)$$

namely,

$$\frac{d\vartheta_n(\xi)}{a\vartheta_n^{2-m} + b\vartheta_n + c\vartheta_n^m} = \frac{d\vartheta_{n-1}(\xi)}{a\vartheta_{n-1}^{2-m} + b\vartheta_{n-1} + c\vartheta_{n-1}^m}. \quad (17)$$

Integrating equation (17) with respect to ξ leads to

$$\vartheta_n(\xi) = \left(\frac{-cA_1 + aA_2(\vartheta_{n-1}(\xi))^{1-m}}{bA_1 + aA_2 + aA_1(\vartheta_{n-1}(\xi))^{1-m}}\right)^{\frac{1}{1-m}}, \quad (18)$$

where A_1 and A_2 are arbitrary constants. With equation (18), we can obtain the solution of equation (2) and the process is called a *Bäcklund transformation*.

3. Applications

The solution of $(2+1)$-dimensional (CBS) equation of equation (1) is obtained by considering the traveling wave transformation

$$\vartheta(x, y, t) = \vartheta(\xi), \quad \xi = (x + y - vt) \quad (19)$$

and plugging it into equation (1), we obtain the following equation:

$$-v\vartheta'' + 6\vartheta'\vartheta'' + \vartheta^{(4)} = 0. \quad (20)$$

Plugging equations (5)-(7) and their derivatives into (20), setting $m = 0$ and collecting all the coefficients of $\vartheta^i(\xi)$ (for $i = 1, 2, 3, 4, 5, 6$), and then equating each collection to zero, we have the following:

$$\vartheta^1(\xi) : b^3c + 6bc^2 + 8abc^2 - bcv = 0,$$

$$\vartheta^2(\xi) : b^4 + 12b^2c + 22ab^2c + 12ac^2 + 16a^2c^2 - b^2v - 2acv = 0,$$

$$\vartheta^3(\xi) : 6b^3 + 15ab^3 + 36abc + 60a^2bc - 3abv = 0,$$

$$\vartheta^4(\xi) : 24ab^2 + 50a^2b^2 + 24a^2c + 40a^3c - 2a^2v = 0,$$

$$\vartheta^5(\xi) : 330a^2b + 60a^3b = 0,$$

$$\vartheta^6(\xi) : 12a^3 + 24a^4 = 0. \tag{21}$$

Solving the system of algebraic equations of equation (21) leads to

$$a = -\frac{1}{2},$$

$$b = 0,$$

$$v = 6c. \tag{22}$$

With the solutions from equation (22) with equations (8)-(15) and (19), we obtain the solutions of equation (1) as:

The periodic solutions can be obtained as

$$\vartheta_1^{\pm}(x, y, t) = -\sqrt{2}\sqrt{-c}\tan\left[\frac{\sqrt{-c}(J - 6ct + x + y)}{\sqrt{2}}\right], \tag{23}$$

$$\vartheta_2^{\pm}(x, y, t) = \sqrt{2}\sqrt{-c}\cot\left[\frac{\sqrt{-c}(J - 6ct + x + y)}{\sqrt{2}}\right]. \tag{24}$$

The singular and dark optical soliton solutions:

$$\vartheta_3^{\pm}(x, y, t) = \sqrt{2}\sqrt{c}\coth\left[\frac{\sqrt{c}(J - 6ct + x + y)}{\sqrt{2}}\right], \tag{25}$$

$$\vartheta_4^{\pm}(x, y, t) = \sqrt{2}\sqrt{c}\tanh\left[\frac{\sqrt{c}(J - 6ct + x + y)}{\sqrt{2}}\right], \tag{26}$$

and the singular solution:

$$\vartheta_5^{\pm}(x,\ y,\ t) = \frac{2}{J - 6ct + x + y}. \tag{27}$$

Figure 1 presents the periodic singular wave solution, that is $\vartheta_1(x,\ y,\ t)$ of equation (23). We consider the following parameters: $c = -10$; $J = -0.4$; $y = 7$.

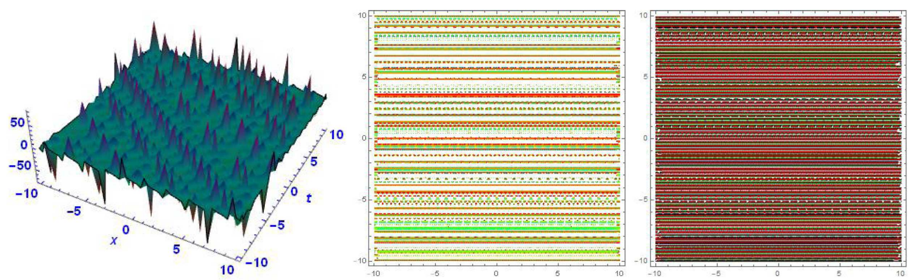

Figure 1. Plot of 3D, density and contour of (23).

Figure 2 presents the periodic singular wave solution, that is $\vartheta_2(x,\ y,\ t)$ of equation (24). We consider the following parameters: $c = 1$; $J = 4$; $y = 2$.

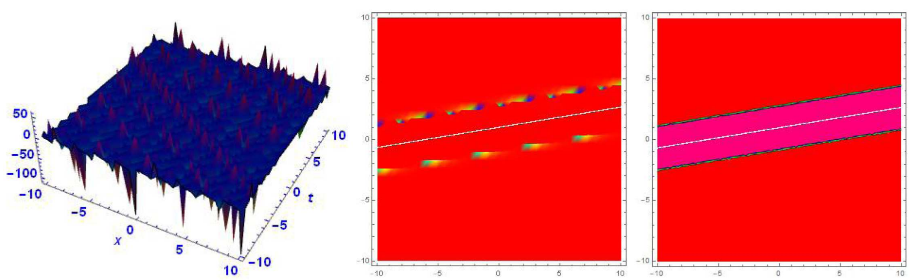

Figure 2. Plot of 3D, density and contour of (24).

Figure 3 presents the dark soliton solution, that is $\vartheta_3(x,\ y,\ t)$ of equation (25). We consider the following parameters: $c = -2.9$; $J = 1$; $y = 2$.

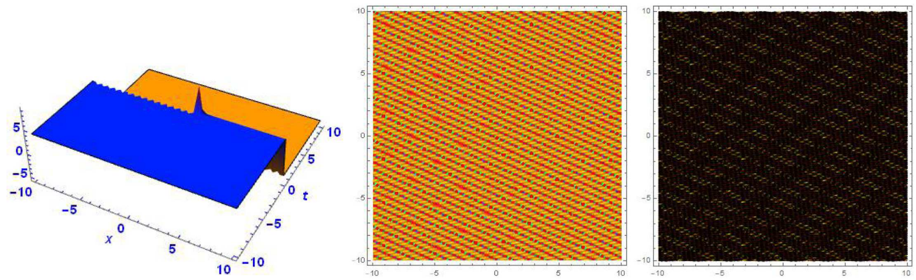

Figure 3. Plot of 3D, density and contour of (25).

Figure 4 presents the singular soliton solution, that is $\vartheta_4(x, y, t)$ of equation (26). We consider the following parameters: $c = 1$; $J = 6$; $y = 7$.

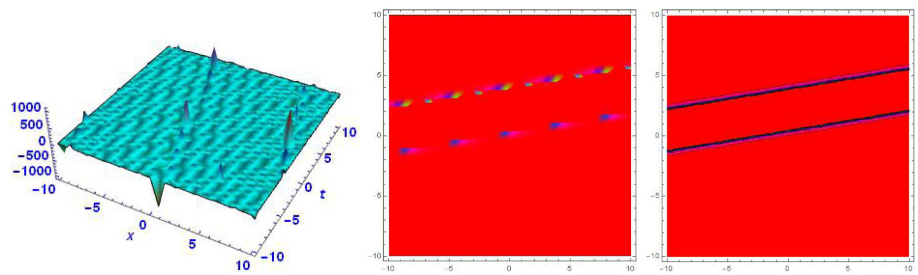

Figure 4. Plot of 3D, density and contour of (26).

Figure 5 presents the singular solution, that is $\vartheta_5(x, y, t)$ of equation (27). We consider the following parameters: $c = -2$; $J = 4$; $y = 0.5$.

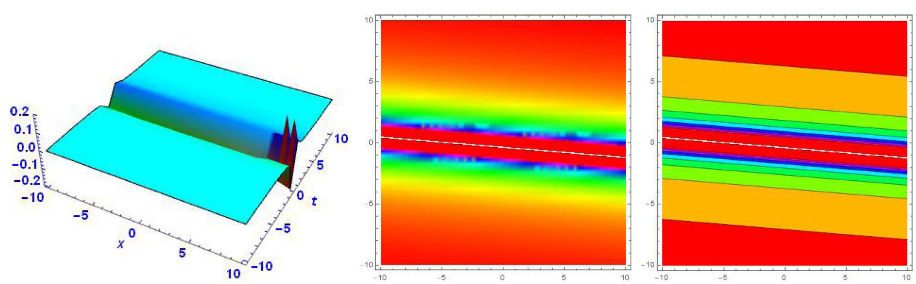

Figure 5. Plot of 3D, density and contour of (27).

4. Concluding Remarks

The RB sub-ODE method was introduced to solve the $(2+1)$ dimensional CBS equation. We retrieved solitary wave solution in the form of kink shape soliton, bright soliton, periodic wave solutions, singular solitons and dark solitons to the $(2+1)$ dimensional CBS equation. The RB sub- ODE method is a powerful and simple mathematical tool that is used for solving complex NLPDEs. The solitary wave results obtained play a vital role in mathematical physics and have unlimited applications in science and technology. The solitary wave results are illustrated by figures.

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