

Research Article

Numerical Solution for Arbitrary Domain of Fractional Integro-differential Equation via the General Shifted Genocchi Polynomials

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The Genocchi polynomial has been increasingly used as a convenient tool to solve some fractional calculus problems, due to their nice properties. However, like some other members in the Appell polynomials, the nice properties are always limited to the interval defined in $[0, 1]$. In this paper, we extend the Genocchi polynomials to the general shifted Genocchi polynomials, $S_n^{(a,b)}(x)$, which are defined for interval $[a, b]$. New properties for this general shifted Genocchi polynomials will be introduced, including the determinant form. This general shifted Genocchi polynomials can overcome the conventional formula of finding the Genocchi coefficients of a function $f(x)$ that involves $f^{(n-1)}(x)$ which may not be defined at $x = 0, 1$. Hence, we use the general shifted Genocchi polynomials to derive the operational matrix and hence to solve the Fredholm-type fractional integro-differential equations with arbitrary domain $[a, b]$.

1. Introduction

The Genocchi polynomials, $G_n(x)$, is one of the members of the Appell polynomials, $A_n(x)$, satisfying the differential relation $(dA_n(x))/dx = nA_{n-1}(x)$, $n = 1, 2, 3, \dots$. Besides, many new results are obtained in the field of number theory and combinatorial [1–4], the Genocchi polynomials are also applied successfully to solve some kind of fractional calculus problems, and its advantages were described in [5–9] mostly via its operational matrix. However, most of the results are applied over the interval $[0, 1]$. Furthermore, for function approximation by using the Genocchi polynomials, the conventional formula of finding the Genocchi coefficients of a function $f(x)$ involves $f^{(n-1)}(x)$ which may not be defined at $x = 0, 1$. To overcome these drawbacks, we propose the general shifted Genocchi polynomials which are more suitable for larger interval $[a, b]$, where $a, b \geq 0$. Hence, in this paper, with the new general shifted Genocchi polynomials, we derive its operational matrix, and then, we solve the frac-

tional integro-differential equation (FIDE) with arbitrary domain, i.e., not limited to interval $[0, 1]$.

FIDE is an equation which contains a fractional derivative term ${}_0D_x^\alpha f(x)$, where α denotes the fractional order derivative with $[\alpha] = n$, i.e., $(n - 1 < \alpha \leq n)$ and an integral kernel operator term $\tilde{K}f(x) = \int K(x, t)f(t)dt$. This paper considers the arbitrary domain FIDE of the 2nd-kind non-homogeneous Fredholm type of the following special class:

$${}_0D_x^\alpha f(x) = h(x) + \lambda \int_a^b K(x, t)f(t)dt, \quad (1)$$
$$f^{(i)}(x_i) = y_i, \quad i = 1, \dots, n,$$

where $f(x)$ is the unknown function to be solved, $K(x, t)$ is the integral kernel, ${}_0D_x^\alpha(x)$ is Caputo's fractional derivative, and $h(x)$ is the nonhomogeneous forced term.

On top of that, solving FIDE is always not an easy task, and reliable numerical methods are needed. Furthermore,

for the Fredholm-type problems, the existing numerical methods are mostly applicable for interval $[0, 1]$. Some of the early published works which for FIDE in $[0, 1]$ are including the collocation method via polynomial spline function [10], the fractional differential transform method [11], and the Taylor expansion method [12]. In this research direction, some researches focus on solving the special class of FIDE, which includes solving fractional partial integro-differential equations by the resolvent kernel method, the Laplace transform [13], and the Laguerre polynomial [14], solving fractional integro-differential equations via the fractional-order Euler polynomials [15] and the Jacobi wavelets [16], solving fourth-order time FIDE with a weakly singular kernel by compact finite difference scheme [17] and nonlinear time-fractional partial integro-differential equation by finite difference scheme [18], and solving nonlinear two-dimensional fractional integro-differential equations via hybrid function [19]. However, the FIDE with the arbitrary domain is relatively less concerned by researchers, and so far, the successful methods applied to this type of problem are limited to the Chebyshev wavelet method [20]. For more methods as well as theories of FIDE/FDE, we refer the readers to some well-known books such as [21, 22].

The rest of the paper is organized as follows: Section 2 is devoted to preliminary results including basic definition, properties and determinant form of the general shifted Genocchi polynomials, $S_n^{(a,b)}(x)$, function approximation by $S_n^{(a,b)}(x)$, and theorem for the analytical expression of the integral of the product of the two general shifted Genocchi polynomials $T_{(n,m)} = \int_a^b S_n^{(a,b)}(x)S_m^{(a,b)}(x)dx$. Section 3 is the main result of this paper, which includes the derivation of a new operational matrix associated with the general shifted Genocchi polynomials. Besides that, the procedure of approximating the integral kernel in terms of the general shifted Genocchi polynomials and analytical expression of the kernel matrix is also explained in this section. Sections 4 and 5 are devoted to the procedure used in this paper and some numerical examples. Section 6 is the conclusion of this paper.

2. Preliminary Results

Here, first, we recall the function approximation by the original Genocchi polynomials. For this purpose, we may approximate a continuous function $f(x)$ in interval $[0, 1]$ in terms of the Genocchi polynomials, $G_n(x)$, as the basis [23, 24] as follows:

$$f(x) = \sum_{n=1}^{\infty} c_n G_n(x), \tag{2}$$

where $G_n(x)$ are the Genocchi polynomials and the Genocchi coefficients are denoted by c_n . But normally, this process is done by using the truncated Genocchi series as follows:

$$f(x) \approx \sum_{n=1}^N c_n G_n(x), \tag{3}$$

where in matrix notation,

$$f(x) = \mathbf{C}^T \mathbf{G}(x), \tag{4}$$

where $\mathbf{C} = [c_1, c_2, \dots, c_N]^T$ is the Genocchi coefficient matrix and $\mathbf{G}(x) = [G_1(x), G_2(x), \dots, G_N(x)]^T$ is the Genocchi basis matrix. The Genocchi coefficients, c_n , can be calculated as

$$c_n = \frac{1}{2n!} \left(f^{(n-1)}(0) + f^{(n-1)}(1) \right), n = 1, 2, \dots, N. \tag{5}$$

2.1. General Shifted Genocchi Polynomials: Definitions and Basic Properties. Equation (5) fails to work for functions that are not $(n - 1)$ -differentiable at the points $x = 0$ or $x = 1$. An example is given as follows where the coefficient, c_3 , is undefined:

Let $N = 3$, $x^{3/2} \approx \sum_{n=1}^3 c_n G_n(x) = c_1 G_1(x) + c_2 G_2(x) + c_3 G_3(x)$, we obtain

$$\begin{aligned} c_3 &= \frac{1}{2(3!)} \left(\left. \frac{d^2}{dx^2} x^{3/2} \right|_{x=0} + \left. \frac{d^2}{dx^2} x^{3/2} \right|_{x=1} \right) \\ &= \frac{1}{2(3!)} \left(\left. \frac{3}{4\sqrt{x}} \right|_{x=0} + \left. \frac{3}{4\sqrt{x}} \right|_{x=1} \right). \end{aligned} \tag{6}$$

To avoid this problem, we define the general shifted Genocchi polynomials by shifting $G_n(x)$ from the interval $[0, 1]$ to the interval $[a, b]$, $0 \leq a \leq b$, i.e., $S_n(x) = G_n((x - a)/(b - a))$, which results in the following definition:

Definition 1. The general shifted Genocchi polynomials $S_n^{(a,b)}(x)$ of order n is defined over the interval $[a, b]$ as

$$S_n^{(a,b)}(x) = G_n\left(\frac{x-a}{b-a}\right) = \sum_{k=0}^n \binom{n}{k} g_{n-k} \left(\frac{x-a}{b-a}\right)^k = \sum_{r=0}^n \binom{n}{r} s_{n-r}^{(a,b)} x^r, \tag{7}$$

where $s_{n-r}^{(a,b)} = \sum_{k=r}^n \left((-1)^{k-r} (g_{n-k}) \binom{n}{k} \binom{k}{r} \right) / ((b-a)^k \binom{n}{r})$ is the general shifted Genocchi number, and let $S_0^{(a,b)}(x) = 0$.

The generating function for the general shifted Genocchi polynomials can be expressed as

$$\frac{2t}{e^t + 1} e^{((x-a)/(b-a))t} = \sum_{n=0}^{\infty} S_n^{(a,b)} \frac{t^n}{n!}, (|t| < \pi). \tag{8}$$

If we choose $a = 2, b = 4$, the first few terms of the general shifted Genocchi polynomials are

$$\begin{aligned} S_1^{(2,4)}(x) &= 1, \\ S_2^{(2,4)}(x) &= x - 3, \\ S_3^{(2,4)}(x) &= \frac{3}{4}x^2 - \frac{9}{2}x + 6, \\ S_4^{(2,4)}(x) &= \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 9, \\ S_5^{(2,4)}(x) &= \frac{5}{16}x^4 - \frac{15}{4}x^3 + 15x^2 - \frac{45}{2}x + 10. \end{aligned} \tag{9}$$

Figures 1 and 2 show the first few original Genocchi polynomials and the general shifted Genocchi polynomials.

Some of the important properties inherited from the classical Genocchi polynomials are

$$\frac{dS_n^{(a,b)}(x)}{dx} = \frac{n}{b-a} S_{n-1}^{(a,b)}(x), \quad n \geq 1, \tag{10}$$

$$\frac{d^k S_n^{(a,b)}(x)}{dx^k} = \begin{cases} 0, & n \leq k, \\ k! \binom{n}{k} \frac{S_{n-k}^{(a,b)}(x)}{(b-a)^k}, & k, n \in \mathbb{N} \cup \{0\}, \quad n > k, \end{cases} \tag{11}$$

$$S_n^{(a,b)}(a) + S_n^{(a,b)}(b) = G_n(0) + G_n(1) = 0, \quad n > 1. \tag{12}$$

Theorem 2. Given an arbitrary integrable continuous function $f(x) \in C^{N-1}(R)$, it can be approximated in terms of the general shifted Genocchi polynomials $S_n^{(a,b)}(x)$ up to order N (i.e., polynomial degree = $N - 1$) by

$$f(x) \approx \sum_{j=1}^N c_j S_j^{(a,b)}(x) = \mathbf{C}^T \mathbf{S}(x). \tag{13}$$

Then, the general shifted Genocchi coefficient c_j is given by

$$c_j = \frac{(b-a)^{j-1}}{2(j!)} \left(f^{(j-1)}(a) + f^{(j-1)}(b) \right). \tag{14}$$

Proof. Let $f(x) = \sum_{k=1}^N c_k S_k^{(a,b)}(x)$. Using Equations (11) and (12), for $1 \leq j$

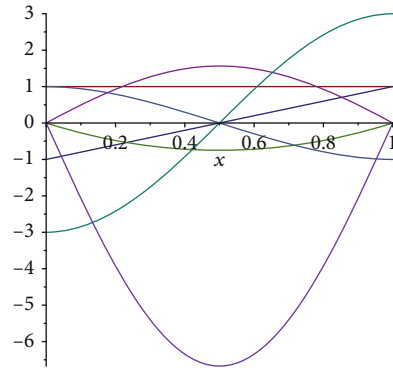


FIGURE 1: The first few original Genocchi polynomials (in interval $[0, 1]$).

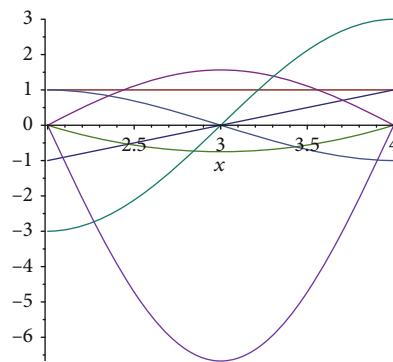


FIGURE 2: The first few general shifted Genocchi polynomials (in interval $[2, 4]$).

$$\begin{aligned} & f^{(j-1)}(a) + f^{(j-1)}(b) \\ &= \sum_{k=1}^N c_k \frac{d^{j-1}}{dx^{j-1}} \left(S_k^{(a,b)}(a) + S_k^{(a,b)}(b) \right) \\ &= \sum_{k=j}^N c_k \frac{(j-1)! \binom{k}{j-1}}{(b-a)^{j-1}} \left(S_{k-(j-1)}^{(a,b)}(a) + S_{k-(j-1)}^{(a,b)}(b) \right) \\ &= c_j \frac{(j-1)! \binom{j}{j-1}}{(b-a)^{j-1}} \left(S_1^{(a,b)}(a) + S_1^{(a,b)}(b) \right) \\ &\quad + \sum_{k=j+1}^N c_k \frac{(j-1)! \binom{k}{j-1}}{(b-a)^{j-1}} \left(S_{k-(j-1)}^{(a,b)}(a) + S_{k-(j-1)}^{(a,b)}(b) \right) \\ &= 2c_j \frac{(j!)}{(b-a)^{j-1}}. \end{aligned} \tag{15}$$

Rearrange the above equation, we obtain $c_j = \frac{(b-a)^{j-1}}{2(j!)} \left(f^{(j-1)}(a) + f^{(j-1)}(b) \right)$. \square

2.2. *Determinant Form of the General Shifted Genocchi Polynomial Sequence.* In this subsection, we will explain that this new general shifted Genocchi polynomial sequence can be also expressed in determinant form and recurrence relation, via modifying the work in [25, 26] for the shifted Genocchi polynomial sequence. For this process, we shift the order of the general shifted Genocchi polynomial sequence

from n to $n + 1$; that is, we have $Sh_n^{(a,b)}(x) = S_{n+1}^{(a,b)}(x)$, where $Sh_n^{(a,b)}(x)$ denotes the general shifted Genocchi polynomial sequence.

Lemma 3. *The determinant form of the general shifted Genocchi polynomial sequence, $Sh_n^{(a,b)}(x)$, which $n > 0$, is given by*

$$Sh_n^{(a,b)}(x) = \frac{(-1)^n}{\prod_{i=0}^n s_{i,i}} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ s_{0,0} & s_{1,0} & s_{2,0} & \cdots & s_{n-1,0} & s_{n,0} \\ 0 & s_{1,1} & s_{2,1} & \cdots & s_{n-1,1} & s_{n,1} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & & & & \end{vmatrix}, \tag{16}$$

and the recurrence relation of the general shifted Genocchi polynomial sequence, $Sh_n^{(a,b)}(x)$, can be written as

$$Sh_n^{(a,b)}(x) = \frac{1}{s_{n,n}} \left(x^n - \sum_{j=0}^{n-1} s_{n,j} Sh_j^{(a,b)}(x) \right). \tag{17}$$

In order to obtain values for $s_{i,j}$, we follow Costabile et al.'s method [25], which we summarize as follows:

Step 1. From $S_n^{(a,b)}(x) = \sum_{r=0}^n \binom{n}{r} s_{n-r}^{(a,b)} x^r$, we obtain the general shifted Genocchi number, $s_i^{(a,b)}$; hence, we calculate the lower triangular Toeplitz matrix, \mathbf{T}_S , with entries

$$t_{i,j} = \frac{s_{i+1-j}^{(a,b)}}{(i+1-j)!}. \tag{18}$$

Step 2. Calculate the upper triangular matrix \mathbf{S} via

$$\mathbf{S} = \mathbf{D}_2^{-1} \mathbf{T}_S^{-1} \mathbf{D}_1^{-1}, \tag{19}$$

where $\mathbf{D}_1 = \text{diag} \{ (i+1)! | i = 0, 1, \dots \}$ and $\mathbf{D}_2 = \text{diag} \{ 1/i! | i = 0, 1, \dots \}$. The values for $s_{i,j}$ can be obtained using the entries of \mathbf{S} .

To show the result of the determinant form of the general shifted Genocchi polynomial sequence, we present that $Sh_3^{(2,5)}(x)$ ($a = 2, b = 5, n = 3$) and $Sh_4^{(2,5)}(x)$ ($a = 2, b = 5, n = 4$) are given by

$$Sh_3^{(2,5)}(x) = \frac{(-1)^3}{1/4!} \begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & \frac{7}{6} & \frac{29}{18} & \frac{133}{54} \\ 0 & \frac{1}{2} & \frac{7}{6} & \frac{29}{12} \\ 0 & 0 & \frac{1}{3} & \frac{7}{6} \end{vmatrix}, \tag{20}$$

$$Sh_4^{(2,5)}(x) = \frac{(-1)^4}{1/5!} \begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & \frac{7}{6} & \frac{29}{18} & \frac{133}{54} & \frac{641}{162} \\ 0 & \frac{1}{2} & \frac{7}{6} & \frac{29}{12} & \frac{133}{27} \\ 0 & 0 & \frac{1}{3} & \frac{7}{6} & \frac{29}{9} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{7}{6} \end{vmatrix}.$$

Hence, the determinant form of the above general shifted Genocchi polynomial sequence is as follows:

$$S_4^{(2,5)}(x) = Sh_3^{(2,5)}(x) = \frac{4}{27}x^3 - \frac{14}{9}x^2 + \frac{40}{9}x - \frac{77}{27},$$

$$S_5^{(2,5)}(x) = Sh_4^{(2,5)}(x) = \frac{5}{81}x^4 - \frac{70}{81}x^3 + \frac{100}{27}x^2 - \frac{385}{81}x + \frac{50}{81}. \tag{21}$$

2.3. *Integral of Product of the General Shifted Genocchi Polynomials.* In this subsection, we derive the analytical expression for the integral of the product of the two general

shifted Genocchi polynomials which will be used frequently in the later parts of the paper. This results in the following theorem:

Theorem 4. Given any two general shifted Genocchi polynomials $S_n^{(a,b)}(x), S_m^{(a,b)}(x)$, for $0 \leq x$

$$\begin{aligned} \gamma_{n,m}^{(a,b)}(x) &= \int S_n^{(a,b)}(x) S_m^{(a,b)}(x) dx = \int_0^x S_n^{(a,b)}(x) S_m^{(a,b)}(x) dx \\ &= \sum_{r=0}^{n-1} (-1)^r (b-a) \frac{n_{(r)}}{(m+1)^{(r+1)}} \\ &\quad \cdot \left(S_{n-r}^{(a,b)}(x) S_{m+1+r}^{(a,b)}(x) - S_{n-r}^{(a,b)}(0) S_{m+1+r}^{(a,b)}(0) \right), \end{aligned} \tag{22}$$

where $n_{(r)}$ is the falling factorial and $(m+1)^{(r+1)}$ is the rising factorial. For $[a, b], 0 \leq a \leq b$, we have

$$\begin{aligned} \gamma_{n,m}^{(a,b)} &= \gamma_{n,m}^{(a,b)}(b) - \gamma_{n,m}^{(a,b)}(a) = \int_a^b S_n^{(a,b)}(x) S_m^{(a,b)}(x) dx \\ &= \sum_{r=0}^{n-1} (-1)^r (b-a) \frac{n_{(r)}}{(m+1)^{(r+1)}} \\ &\quad \cdot \left(S_{n-r}^{(a,b)}(b) S_{m+1+r}^{(a,b)}(b) - S_{n-r}^{(a,b)}(a) S_{m+1+r}^{(a,b)}(a) \right) \tag{23} \\ &= \sum_{r=0}^{n-1} (-1)^r (b-a) \frac{n_{(r)}}{(m+1)^{(r+1)}} \\ &\quad \cdot \left(S_{n-r}^{(a,b)}(b) S_{m+1+r}^{(a,b)}(b) - g_{n-r} g_{m+1+r} \right), \end{aligned}$$

where $g_k = S_k^{(a,b)}(a) = G_k(0)$.

Proof. In order to prove this theorem, we need the following expression:

$$\begin{aligned} \frac{dS_m^{(a,b)}(x)}{dx} &= \frac{m}{b-a} S_{m-1}^{(a,b)}(x), \tag{24} \\ \int_0^x S_m^{(a,b)}(x) dx &= \frac{b-a}{m+1} \left(S_{m+1}^{(a,b)}(x) - S_{m+1}^{(a,b)}(0) \right). \end{aligned}$$

For

$$\gamma_{n,m}^{(a,b)}(x) = \int_0^x S_n^{(a,b)}(x) S_m^{(a,b)}(x) dx, \tag{25}$$

and hence, using integration by parts, we have

$$\begin{aligned} \gamma_{n,m}^{(a,b)}(x) &= S_n^{(a,b)}(x) \left(\frac{(b-a) \left(S_{m+1}^{(a,b)}(x) - S_{m+1}^{(a,b)}(0) \right)}{m+1} \right) \\ &\quad - \int_0^x \left(\frac{n}{b-a} S_{n-1}^{(a,b)}(x) \right) \frac{(b-a) \left(S_{m+1}^{(a,b)}(x) - S_{m+1}^{(a,b)}(0) \right)}{m+1} dx \\ &= S_n^{(a,b)}(x) \left(\frac{(b-a) \left(S_{m+1}^{(a,b)}(x) - S_{m+1}^{(a,b)}(0) \right)}{m+1} \right) \\ &\quad - \frac{n}{m+1} \int_0^x S_{n-1}^{(a,b)}(x) S_{m+1}^{(a,b)}(x) dx \\ &\quad + \frac{n}{m+1} S_{m+1}^{(a,b)}(0) \int_0^x S_{n-1}^{(a,b)}(x) dx \\ &= \frac{b-a}{m+1} S_n^{(a,b)}(x) \left(S_{m+1}^{(a,b)}(x) - S_{m+1}^{(a,b)}(0) \right) \\ &\quad - \frac{n}{m+1} \int_0^x S_{n-1}^{(a,b)}(x) S_{m+1}^{(a,b)}(x) dx \\ &\quad + \frac{b-a}{m+1} S_{m+1}^{(a,b)}(0) \left(S_n^{(a,b)}(x) - S_n^{(a,b)}(0) \right) \\ &= \frac{b-a}{m+1} \left(S_n^{(a,b)}(x) S_{m+1}^{(a,b)}(x) - S_n^{(a,b)}(0) S_{m+1}^{(a,b)}(0) \right) \\ &\quad - \frac{n}{m+1} \int_0^x S_{n-1}^{(a,b)}(x) S_{m+1}^{(a,b)}(x) dx. \end{aligned} \tag{26}$$

By using Equation (25), we obtain

$$\begin{aligned} \gamma_{n,m}^{(a,b)}(x) &= (b-a) \left(\frac{S_n^{(a,b)}(x) S_{m+1}^{(a,b)}(x) - S_n^{(a,b)}(0) S_{m+1}^{(a,b)}(0)}{m+1} \right) \\ &\quad - \frac{n}{m+1} \gamma_{n-1,m+1}^{(a,b)}(x) \\ &= (b-a) \left(\frac{S_n^{(a,b)}(x) S_{m+1}^{(a,b)}(x) - S_n^{(a,b)}(0) S_{m+1}^{(a,b)}(0)}{m+1} \right) \\ &\quad - \frac{n}{m+1} \left((b-a) \left(\frac{S_{n-1}^{(a,b)}(x) S_{m+2}^{(a,b)}(x) - S_{n-1}^{(a,b)}(0) S_{m+2}^{(a,b)}(0)}{m+2} \right) \right. \\ &\quad \left. - \frac{n-1}{m+2} \gamma_{n-2,m+2}^{(a,b)}(x) \right) \\ &= (b-a) \left(\frac{S_n^{(a,b)}(x) S_{m+1}^{(a,b)}(x) - S_n^{(a,b)}(0) S_{m+1}^{(a,b)}(0)}{m+1} \right) \\ &\quad + \frac{(-1)^1 (b-a)n}{m+1} \left(\frac{S_{n-1}^{(a,b)}(x) S_{m+2}^{(a,b)}(x) - S_{n-1}^{(a,b)}(0) S_{m+2}^{(a,b)}(0)}{m+2} \right) \\ &\quad + \frac{(-1)^2 n(n-1)}{(m+1)(m+2)} \gamma_{n-2,m+2}^{(a,b)}(x) = \dots \end{aligned} \tag{27}$$

These processes continue recursively for n times, and then, we will obtain

$$\begin{aligned}
 &= \dots = \sum_{r=0}^{n-1} (-1)^r (b-a) \frac{n(n-1) \dots (n-r+1)}{(m+1) \dots (m+r)} \\
 &\quad \cdot \left(S_{n-r}^{(a,b)}(x) S_{m+1+r}^{(a,b)}(x) - S_{n-r}^{(a,b)}(0) S_{m+1+r}^{(a,b)}(0) \right) \\
 &= \sum_{r=0}^{n-1} (-1)^r (b-a) \frac{n_r}{(m+1)^{(r+1)}} \\
 &\quad \cdot \left(S_{n-r}^{(a,b)}(x) S_{m+1+r}^{(a,b)}(x) - S_{n-r}^{(a,b)}(0) S_{m+1+r}^{(a,b)}(0) \right).
 \end{aligned} \tag{28}$$

□

3. Main Result

3.1. General Shifted Genocchi Polynomial Operational Matrix of Fractional Derivative. In this section, we will derive the analytical expression of the general shifted Genoc-

chi polynomial operational matrix of fractional derivative in the Caputo sense, which is the $N \times N$ matrix ${}_0\mathbf{P}_S^\alpha$, where

$$\begin{aligned}
 &{}_0D_x^\alpha \mathbf{S}(x) = {}_0\mathbf{P}_S^\alpha \mathbf{S}(x), \\
 &{}_0D_x^\alpha \begin{bmatrix} S_1^{(a,b)} \\ S_2^{(a,b)} \\ \vdots \\ S_N^{(a,b)} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2N} \\ \vdots & \dots & \dots & \vdots \\ \rho_{N1} & \rho_{N2} & \dots & \rho_{NN} \end{bmatrix} \begin{bmatrix} S_1^{(a,b)} \\ S_2^{(a,b)} \\ \vdots \\ S_N^{(a,b)} \end{bmatrix}.
 \end{aligned} \tag{29}$$

To derive the ${}_0\mathbf{P}_S^\alpha$, we first prove the following Lemma 5.

Lemma 5. Caputo's fractional derivative of fractional order α of a general shifted Genocchi polynomial, $S_i^{(a,b)}(x)$, of order i is given by

$${}_0D_x^\alpha S_i^{(a,b)}(x) = \begin{cases} \sum_{r=\lceil \alpha \rceil}^i \sum_{k=0}^{r-\lceil \alpha \rceil} \frac{(-a)^{r-\lceil \alpha \rceil-k} i! g_{i-r} x^{\lceil \alpha \rceil - \alpha + k}}{(b-a)^r (i-r)! (r-\lceil \alpha \rceil - k)! \Gamma(\lceil \alpha \rceil - \alpha + k + 1)}, & n-1 < \alpha \leq n, n \in \mathbb{N}, \quad i \geq \alpha \\ 0, & i < \alpha. \end{cases} \tag{30}$$

Proof. For $n-1 < \alpha \leq n, n = \lceil \alpha \rceil$

$$\begin{aligned}
 &{}_0D_x^\alpha S_i^{(a,b)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} G_i \left(\frac{t-a}{b-a} \right) dt \\
 &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} \sum_{r=0}^i \binom{i}{r} g_{i-r} \left(\frac{t-a}{b-a} \right)^r dt \\
 &= \frac{1}{\Gamma(n-\alpha)} \sum_{r=n}^i \frac{\binom{i}{r} g_{i-r}}{(b-a)^r} n! \binom{r}{n} \\
 &\quad \cdot \int_0^x (x-t)^{n-\alpha-1} (t-a)^{r-n} dt \\
 &= \frac{1}{\Gamma(n-\alpha)} \sum_{r=n}^i \frac{\binom{i}{r} g_{i-r}}{(b-a)^r} n! \binom{r}{n} \\
 &\quad \cdot \int_0^x (x-t)^{n-\alpha-1} \sum_{k=0}^{r-n} \binom{r-n}{k} t^k (-a)^{r-n-k} dt \\
 &= \frac{1}{\Gamma(n-\alpha)} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} \binom{i}{r} \binom{r-n}{k} g_{i-r}}{(b-a)^r} n! \\
 &\quad \cdot \binom{r}{n} \int_0^x (x-t)^{n-\alpha-1} t^k dt.
 \end{aligned} \tag{31}$$

Substitute $t = xu$,

$$\begin{aligned}
 &{}_0D_x^\alpha S_i^{(a,b)}(x) = \frac{1}{\Gamma(n-\alpha)} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} \binom{i}{r} \binom{r-n}{k} g_{i-r}}{(b-a)^r} n! \\
 &\quad \cdot \binom{r}{n} x^{n-\alpha+k} \int_0^1 (1-u)^{n-\alpha-1} u^k du \\
 &= \frac{1}{\Gamma(n-\alpha)} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} \binom{i}{r} \binom{r-n}{k} g_{i-r}}{(b-a)^r} n! \\
 &\quad \cdot \binom{r}{n} x^{n-\alpha+k} B(k+1, n-\alpha) = \frac{1}{\Gamma(n-\alpha)} \\
 &\quad \cdot \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} \binom{i}{r} \binom{r-n}{k} g_{i-r}}{(b-a)^r} n! \binom{r}{n} x^{n-\alpha+k} \\
 &\quad \cdot \frac{\Gamma(k+1) \Gamma(n-\alpha)}{\Gamma(n-\alpha+k+1)} \\
 &= \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} i! g_{i-r}}{(b-a)^r (i-r)! (r-n-k)! \Gamma(n-\alpha+k+1)} x^{n-\alpha+k},
 \end{aligned} \tag{32}$$

where $B(x, y)$ is the beta function which can be found using $\int_0^1 u^{x-1} (1-u)^{y-1} du$ for $\text{Re}(x), \text{Re}(y) > 0$. □

Hence, we obtain the theorem for operational matrix ${}_0\mathbf{P}_S^\alpha$ as follows:

Theorem 6. Given a set $S_i^{(a,b)}(x), i = 1, \dots, N$ of the N general shifted Genocchi polynomials, the general shifted Genocchi polynomial operational matrix of fractional derivative in the Caputo sense of order α over the interval $[0, 1]$ is the $N \times N$ matrix ${}_0\mathbf{P}_S^\alpha$, given by

$${}_0\mathbf{P}_S^\alpha = M^{(\alpha)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,1} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,2} & \dots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \rho_{\lceil\alpha\rceil,N} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^i \rho_{i,1} & \sum_{k=\lceil\alpha\rceil}^i \rho_{i,2} & \dots & \sum_{k=\lceil\alpha\rceil}^i \rho_{i,N} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^N \rho_{N,1} & \sum_{k=\lceil\alpha\rceil}^N \rho_{N,2} & \dots & \sum_{k=\lceil\alpha\rceil}^N \rho_{N,N} \end{pmatrix}, \tag{33}$$

where $\rho_{i,j}$ is given by

$$\rho_{i,j} = \frac{(b-a)^{j-1}}{2(j!)} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} i! g_{i-r} (a^{n-\alpha+k-j+1} + b^{n-\alpha+k-j+1})}{(b-a)^r (i-r)! (r-n-k)! \Gamma(n-\alpha+k-j+2)}. \tag{34}$$

Proof. Let ${}_0D_x^\alpha S_i^{(a,b)}(x) = \sum_{j=1}^N \rho_{ij} S_j^{(a,b)}(x)$. Then, using the function approximation as in Theorem 2 and Caputo's fractional derivative for $S_i^{(a,b)}(x)$ in Lemma 5, we have

$$\begin{aligned} \rho_{ij} &= \frac{(b-a)^{j-1}}{2(j!)} \left(\frac{d^{j-1}}{dx^{j-1}} {}_0D_x^\alpha S_i^{(a,b)}(x) \Big|_{x=a} + \frac{d^{j-1}}{dx^{j-1}} {}_0D_x^\alpha S_i^{(a,b)}(x) \Big|_{x=b} \right) \\ &= \frac{(b-a)^{j-1}}{2(j!)} \left(\frac{d^{j-1}}{dx^{j-1}} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} i! g_{i-r} x^{n-\alpha+k}}{(b-a)^r (i-r)! (r-n-k)! \Gamma(n-\alpha+k+1)} \Big|_{x=a} \right. \\ &\quad \left. + \frac{d^{j-1}}{dx^{j-1}} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} i! g_{i-r} x^{n-\alpha+k}}{(b-a)^r (i-r)! (r-n-k)! \Gamma(n-\alpha+k+1)} \Big|_{x=b} \right) \\ &= \frac{(b-a)^{j-1}}{2(j!)} \times \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} i! g_{i-r} (j-1)! \binom{n-\alpha+k}{j-1} (x^{n-\alpha+k-j+1} \Big|_{x=a} + x^{n-\alpha+k-j+1} \Big|_{x=b})}{(b-a)^r (i-r)! (r-n-k)! \Gamma(n-\alpha+k+1)} \\ &= \frac{(b-a)^{j-1}}{2(j!)} \sum_{r=n}^i \sum_{k=0}^{r-n} \frac{(-a)^{r-n-k} i! g_{i-r} (a^{n-\alpha+k-j+1} + b^{n-\alpha+k-j+1})}{(b-a)^r (i-r)! (r-n-k)! \Gamma(n-\alpha+k-j+2)}. \end{aligned} \tag{35}$$

□

As the formula given in Theorem 6 may fail to work for $a = 0$, we use the matrix approach to derive the ${}_0\mathbf{P}_S^\alpha$ as given in Theorem 8. To prove Theorem 8, we need the following lemma:

Lemma 7. The matrix Θ_N defined as $\Theta_N = \int_0^1 ({}_0D_x^\alpha S(x)) * S^T(x) dx$ is given by

$$\Theta_N = [\theta_{ik}]_{N \times N} = \left[\int_0^1 {}_0D_x^\alpha G_i(x) G_k(x) dx \right]_{N \times N}, \tag{36}$$

where

$$\theta_{ik} = \sum_{r=\lceil\alpha\rceil}^i \sum_{p=0}^k \frac{i! \binom{k}{p} g_{i-r} g_{k-p}}{(i-r)! (r-\alpha+p+1) \Gamma(r-\alpha+1)}. \tag{37}$$

Proof. From Caputo's fractional derivative of the Genocchi polynomials [5, 23, 24], we have

$$\begin{aligned} {}_0D_x^\alpha G_i(x) &= \sum_{r=\lceil\alpha\rceil}^i \frac{i! g_{i-r}}{(i-r)! \Gamma(r-\alpha+1)} x^{r-\alpha} \\ {}_0D_x^\alpha G_i(x) G_k(x) &= \left(\sum_{r=\lceil\alpha\rceil}^i \frac{i! g_{i-r}}{(i-r)! \Gamma(r-\alpha+1)} x^{r-\alpha} \right) \left(\sum_{p=0}^k \binom{k}{p} g_{k-p} x^p \right) \\ &= \sum_{r=\lceil\alpha\rceil}^i \sum_{p=0}^k \frac{i! g_{i-r}}{(i-r)! \Gamma(r-\alpha+1)} \binom{k}{p} g_{k-p} x^{r-\alpha+p}. \end{aligned} \tag{38}$$

Integrate both sides, and we obtain

$$\begin{aligned} \int_0^1 {}_0D_x^\alpha G_i(x) G_k(x) dx &= \sum_{r=\lceil\alpha\rceil}^i \sum_{p=0}^k \frac{i! g_{i-r}}{(i-r)! \Gamma(r-\alpha+1)} \binom{k}{p} g_{k-p} \int_0^1 x^{r-\alpha+p} dx \\ &= \sum_{r=\lceil\alpha\rceil}^i \sum_{p=0}^k \frac{i! \binom{k}{p} g_{i-r} g_{k-p}}{(i-r)! (r-\alpha+p+1) \Gamma(r-\alpha+1)}. \end{aligned} \tag{39}$$

□

Theorem 8. Given a set $S_i^{(a,b)}(x), i = 1, \dots, N$ of the N general shifted Genocchi polynomials, the general shifted Genocchi polynomial operational matrix of Caputo's fractional derivative of order α over the interval $[a, b]$ is the $N \times N$ matrix ${}_0\mathbf{P}_S^\alpha$, given by

$${}_0\mathbf{P}_S^\alpha = \Theta_N \mathbf{T}^{(a,b)^{-1}}, \tag{40}$$

where Θ_N is given in Lemma 7 and $\mathbf{T}^{(a,b)} = [\gamma_{nm}^{(a,b)}]_{N \times N}$ with elements $\gamma_{nm}^{(a,b)}$ is given in Theorem 4.

Proof. From Equation (29),

$$\begin{aligned} {}_0D_x^\alpha \mathbf{S}(x) &= {}_0\mathbf{P}_S^\alpha \mathbf{S}(x), \\ ({}_0D_x^\alpha \mathbf{S}(x)) \mathbf{S}^T(x) &= {}_0\mathbf{P}_S^\alpha \mathbf{S}(x) \mathbf{S}^T(x), \\ \int_0^1 ({}_0D_x^\alpha \mathbf{S}(x)) \mathbf{S}^T(x) dx &= {}_0\mathbf{P}_S^\alpha \left(\int_0^1 \mathbf{S}(x) \mathbf{S}^T(x) dx \right), \\ \Theta_N &= {}_0\mathbf{P}_S^{\alpha T} \mathbf{T}_0^{(0,1)}, \\ \mathbf{P}_S^\alpha &= \Theta_N \mathbf{T}^{(0,1)^{-1}}. \end{aligned} \quad (41)$$

□

3.2. Approximation of Integral Kernel by the General Shifted Genocchi Polynomials. Here, we approximate the integral kernel $K(x, t)$ in terms of the truncated series of the general shifted Genocchi polynomials $S_n^{(a,b)}(x)$:

$$\begin{aligned} K(x, t) &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} S_i^{(a,b)}(x) S_j^{(a,b)}(t) \approx \sum_{i=1}^N \sum_{j=1}^N k_{ij} S_i^{(a,b)}(x) S_j^{(a,b)}(t) \\ &= \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t). \end{aligned} \quad (42)$$

We present two approaches for obtaining the kernel matrix \mathbf{K}_S .

3.2.1. Method 1: Conventional Genocchi Coefficient Formula. Using the conventional method of finding the Genocchi coefficients for a single variable function $f(x)$, we can extend the formula to a two-variable function $K(x, t)$ which is continuous and $(N-1)$ differentiable in interval $[0, 1]$. This results in the following theorem:

Theorem 9. *Let $K(x, t)$ be a two-variable continuous function in $C^{N-1}([0, 1])$. Then, $K(x, t)$ can be approximated in terms of the general shifted Genocchi polynomials up to order N , i.e., $K(x, t) \approx \sum_{i=1}^N \sum_{j=1}^N k_{ij} S_i^{(a,b)}(x) S_j^{(a,b)}(t) = \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t)$, $N \in \mathbb{N}$, where \mathbf{S} is the general shifted Genocchi polynomial basis matrix and \mathbf{K}_S is the $N \times N$ integral kernel matrix in the general shifted Genocchi basis given by*

$$\begin{aligned} \mathbf{K}_S &= [k_{ij}]_{N \times N}, \\ k_{ij} &= \frac{(b-a)^{i+j-2}}{4(i!j!)} \left(K_{(x,t)}^{(i-1,j-1)}(a, a) + K_{(x,t)}^{(i-1,j-1)}(a, b) \right. \\ &\quad \left. + K_{(x,t)}^{(i-1,j-1)}(b, a) + K_{(x,t)}^{(i-1,j-1)}(b, b) \right), \end{aligned} \quad (43)$$

where

$$K_{(x,t)}^{(i-1,j-1)}(a, b) = \frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial^{j-1}}{\partial t^{j-1}} K(x, t) \Big|_{x=a, t=b}. \quad (44)$$

Proof. Assume that the kernel $K(x, t)$ is approximated using N number of the general shifted Genocchi polynomials, i.e.,

$$\begin{aligned} K(x, t) &\approx = \sum_{i=1}^N \sum_{j=1}^N k_{ij} S_i^{(a,b)}(x) S_j^{(a,b)}(t) \\ &= \sum_{j=1}^N \left(\sum_{i=1}^N k_{ij} S_i^{(a,b)}(x) \right) S_j^{(a,b)}(t). \end{aligned} \quad (45)$$

Set $\phi_j(x) = \sum_{i=1}^N k_{ij} S_i^{(a,b)}(x)$. Hence, $K(x, t) = \sum_{j=1}^N \phi_j(x) S_j^{(a,b)}(t)$. Using the formula of the general shifted Genocchi coefficients for $S_j^{(a,b)}(t)$ (i.e., Theorem 2 with respect to t variable),

$$\phi_j(x) = \frac{(b-a)^{j-1}}{2(j!)} \left(\frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{t=a} + \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{t=b} \right). \quad (46)$$

Now using the above expression of $\phi_j(x)$ together with the formula of the general shifted Genocchi coefficients for $S_i^{(a,b)}(x)$ (i.e., with respect to x variable) instead, we obtain

$$\begin{aligned} k_{ij} &= \frac{(b-a)^{i-1}}{2(i!)} \left(\frac{\partial^{i-1} \phi_j(x)}{\partial x^{i-1}} \Big|_{x=a} + \frac{\partial^{i-1} \phi_j(x)}{\partial x^{i-1}} \Big|_{x=b} \right) \\ &= \frac{(b-a)^{i-1}}{2(i!)} \left[\frac{\partial^{i-1}}{\partial x^{i-1}} \frac{(b-a)^{j-1}}{2(j!)} \left(\frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{t=a} + \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{t=b} \right) \right] \Big|_{x=a} \\ &\quad + \frac{\partial^{i-1}}{\partial x^{i-1}} \frac{(b-a)^{j-1}}{2(j!)} \left(\frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{t=a} + \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{t=b} \right) \Big|_{x=b} \\ &= \frac{(b-a)^{i-1} (b-a)^{j-1}}{4(i!j!)} \left[\left(\frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{x=a, t=a} \right. \right. \\ &\quad \left. \left. + \frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{x=a, t=b} \right) + \left(\frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{x=b, t=a} \right. \right. \\ &\quad \left. \left. + \frac{\partial^{i-1}}{\partial x^{i-1}} \frac{\partial^{j-1} K(x, t)}{\partial t^{j-1}} \Big|_{x=b, t=b} \right) \right]. \end{aligned} \quad (47)$$

In a similar way, the above approach can be extended to finding the general shifted Genocchi coefficients for the approximation of a multivariable function. □

3.2.2. Method 2: Matrix Method. The classical way will not work for the kernel function not differentiable at $x, t = 0, 1$; then, we can adopt the matrix approach similar to that of finding the general shifted Genocchi coefficients in Theorem 2 to arrive at the following theorem:

Theorem 10. *Let $K(x, t)$ be a two-variable continuous function in $C^{N-1}([a, b])$. Then, $K(x, t)$ can be approximated in terms of the general shifted Genocchi polynomials up to order N , i.e., $K(x, t) \approx \sum_{i=1}^N \sum_{j=1}^N k_{ij} S_i^{(a,b)}(x) S_j^{(a,b)}(t) = \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t)$, $N \in \mathbb{N}$, where \mathbf{S} is the general shifted Genocchi polynomial basis matrix and \mathbf{K}_S is the $N \times N$ integral kernel matrix in the general shifted Genocchi basis given by*

Example 1: Graph of exact solution versus approximate solution

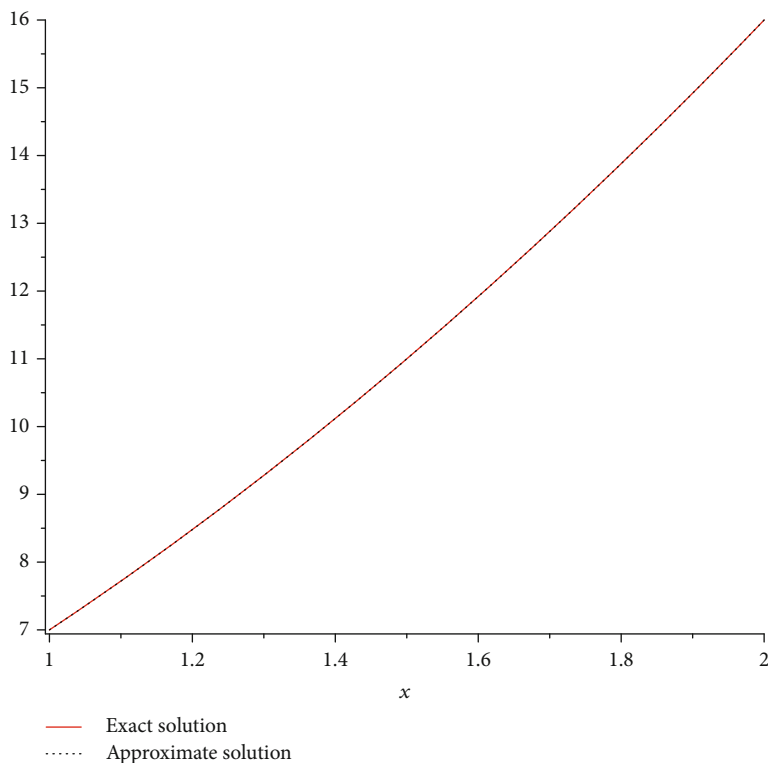


FIGURE 3: Comparison of exact solution $f(x)$ (red colour) and approximate solution $f^*(x)$ (dot line) of Example 1 for $N = 4$.

$$\mathbf{K}_S = \left(\mathbf{T}^{(a,b)}\right)^{-1} \mathbf{N}_S \left(\mathbf{T}^{(a,b)}\right)^{-1}, \tag{48}$$

TABLE 1: Comparison of the approximate solution $f^*(x)$ using $N = 4$ with exact solution $f(x) = 2x^2 + 3x + 2$ and absolute errors for Example 1.

t	Exact sol. $f(x)$	Approx. sol. $f^*(x)$	Abs. error $ f(x) - f^*(x) $
1.0	7.0000000000	7.0000000000	0
1.1	7.7200000000	7.7200000000	0
1.2	8.4800000000	8.4800000000	0
1.3	9.2800000000	9.2800000000	0
1.4	10.1200000000	10.1200000000	0
1.5	11.0000000000	11.0000000000	0
1.6	11.9200000000	11.9200000000	0
1.7	12.8800000000	12.8800000000	0
1.8	13.8800000000	13.8800000000	0
1.9	14.9200000000	14.9200000000	0
2.0	16.0000000000	16.0000000000	0

where

$$\begin{aligned} \mathbf{N}_S &= [\eta_{pq}]_{N \times N} = \left[\int_a^b \int_a^b K(x, t) S_p^{(a,b)}(x) S_q^{(a,b)}(t) dx dt \right]_{N \times N}, \\ \left(\mathbf{T}^{(a,b)}\right)^{-1} &= \left([\gamma_{nm}^{(a,b)}]_{N \times N} \right)^{-1} \\ &= \left(\left[\sum_{r=0}^{n-1} \frac{(-1)^r (b-a)^{n(r)}}{(m+1)^{(r+1)}} \left(S_{n-r}^{(a,b)}(b) S_{m+1+r}^{(a,b)}(b) - g_{n-r} g_{m+1+r} \right) \right]_{N \times N} \right)^{-1}. \end{aligned} \tag{49}$$

Proof. From, $K(x, t) \approx \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t)$,

$$\begin{aligned} \mathbf{S}(x) K(x, t) &= \mathbf{S}(x) \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t), \\ \int_a^b \mathbf{S}(x) K(x, t) dx &= \left(\int_a^b \mathbf{S}(x) \mathbf{S}^T(x) dx \right) \mathbf{K}_S \mathbf{S}(t) \\ &= \mathbf{T}^{(a,b)} \mathbf{K}_S \mathbf{S}(t), \end{aligned}$$

$$\begin{aligned} \left(\int_a^b \mathbf{S}(x) K(x, t) dx \right) \mathbf{S}^T(t) &= \mathbf{T}^{(a,b)} \mathbf{K}_S \mathbf{S}(t) \mathbf{S}^T(t), \\ \int_a^b \int_a^b K(x, t) \mathbf{S}(x) \mathbf{S}^T(t) dx dt &= \mathbf{T}^{(a,b)} \mathbf{K}_S \left(\int_a^b \mathbf{S}(t) \mathbf{S}^T(t) dt \right), \end{aligned} \tag{50}$$

where $\mathbf{T}^{(a,b)} = \int_a^b \mathbf{S}(x) \mathbf{S}^T(x) dx = \int_a^b \mathbf{S}(t) \mathbf{S}^T(t) dt$. Define $\mathbf{N}_S = \int_a^b \int_a^b K(x, t) \mathbf{S}(x) \mathbf{S}^T(t) dx dt$. Thus,

$$\begin{aligned} \mathbf{N}_S &= \mathbf{T}^{(a,b)} \mathbf{K}_S \mathbf{T}^{(a,b)}, \\ \mathbf{K}_S &= \left(\mathbf{T}^{(a,b)}\right)^{-1} \mathbf{N}_S \left(\mathbf{T}^{(a,b)}\right)^{-1}. \end{aligned} \tag{51}$$

□

Example 2: Graph of exact solution versus approximate solution for $N = 4$ over interval $[2,4]$

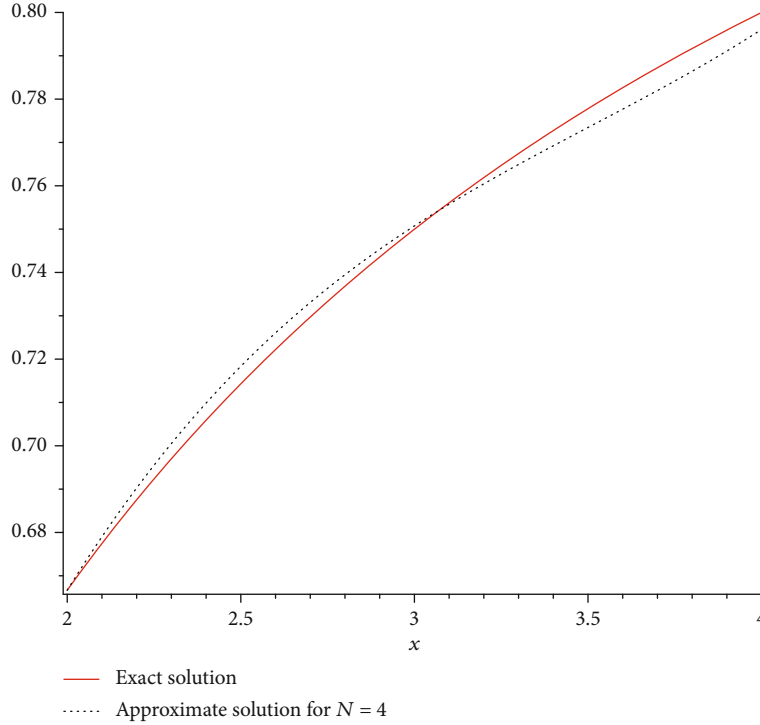


FIGURE 4: Comparison of exact solution $f(x)$ (red colour) and approximate solution $f^*(x)$ (dot line) of Example 2 for $N = 4$.

4. Procedure of Solving Arbitrary Domain Fractional Integro-differential Equation

We will adopt the same approach as the other operational matrix methods for solving FIDE by approximating each term in the equation by the general shifted Genocchi polynomials. From the following class of the 2nd-kind Fredholm FIDE,

$${}_0D_x^\alpha f(x) = h(x) + \lambda \int_a^b K(x, t) f(t) dt, \quad (52)$$

$$f^{(i)}(x_i) = y_i, \quad i = 1, \dots, m. \quad (53)$$

Now, we approximate each term in Equation (52) with the corresponding approximation using the N order of the general shifted Genocchi polynomials,

$$\begin{aligned} {}_0D_x^\alpha f(x) &\approx \mathbf{C}_0^T \mathbf{P}_S^\alpha \mathbf{S}(x), \\ h(x) &\approx \mathbf{H}^T \mathbf{S}(x), \\ K(x, t) &\approx \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t). \end{aligned} \quad (54)$$

Hence, FIDE in Equation (52) in matrix form becomes:

$$\begin{aligned} \mathbf{S}^T(x) ({}_0\mathbf{P}_S^\alpha)^T \mathbf{C} &= \mathbf{S}^T(x) \mathbf{H} + \lambda \int_a^b \mathbf{S}^T(x) \mathbf{K}_S \mathbf{S}(t) \mathbf{S}^T(t) \mathbf{C} dt, \\ \mathbf{S}^T(x) ({}_0\mathbf{P}_S^\alpha)^T \mathbf{C} &= \mathbf{S}^T(x) \mathbf{H} + \lambda \mathbf{S}^T(x) \mathbf{K}_S \left(\int_a^b \mathbf{S}(t) \mathbf{S}^T(t) dt \right) \mathbf{C}, \\ \mathbf{S}^T(x) \left(({}_0\mathbf{P}_S^\alpha)^T \mathbf{C} - \mathbf{H} - \lambda \mathbf{K}_S \mathbf{T}^{(a,b)} \mathbf{C} \right) &= \mathbf{0}, \end{aligned} \quad (55)$$

where $\mathbf{T}^{(a,b)} = \int_a^b \mathbf{S}(t) \mathbf{S}^T(t) dt = [\gamma_{nm}^{(a,b)}]_{N \times N}$. Then, we select the N equally spaced points within the interval $[a, b]$, i.e., $v_r = a + ((b-a)(r-1))/(N-1)$, $r = 1, \dots, N$ as the collocation points and substituting these into Equation (55), and we obtain a system of $N+1$ algebraic equations in terms of the N general shifted Genocchi coefficients $\mathbf{C} = [c_1 \dots c_N]^T$. Choosing any $N-1$ equations from the above system together with the initial conditions, $f(x_0) = \mathbf{S}^T(x) \mathbf{C} = y_0$, we can solve for the general shifted Genocchi coefficients $\mathbf{C} = [c_1 \dots c_N]^T$ of the unknown function $f(x)$. The approximate solution $f^*(x) = \mathbf{C}^T \mathbf{S}(x)$ will be compared to the exact solution $f(x)$ over the interval $[a, b]$.

5. Numerical Examples

Here are two examples of FIDE of Equation (1) which are solved using the proposed method via the general shifted Genocchi polynomials and its operational matrix. The computation was done via Maple software.

Example 1. Consider the following FIDE:

$${}_0D_x^{(5/3)} f(x) = 6x^{1/3} \frac{\sqrt{3}\Gamma(2/3)}{\pi} - \frac{471}{10}x^2 + \int_1^2 x^2 t^3 f(t) dt. \quad (56)$$

Let $f(1) = 7$, the exact solution is $f(x) = 2x^2 + 3x + 2$. Since $[a, b] = [1, 2]$, we use the general shifted Genocchi

TABLE 2: Comparison of the approximate solution $f^*(x)$ using $N = 4$ with exact solution $f(x) = x/(x + 1)$ and absolute errors for Example 2.

t	Exact sol. $f(x)$	Approx. sol. $f^*(x)$	Abs. error $ f(x) - f^*(x) $
2.0	0.6666666667	0.6666666670	$3.00000E - 10$
2.1	0.6774193548	0.6790021440	$1.58279E - 03$
2.2	0.6875000000	0.6902602348	$2.76023E - 03$
2.3	0.6969696970	0.7005176063	$3.54791E - 03$
2.4	0.7058823529	0.7098509254	$3.96857E - 03$
2.5	0.7142857143	0.7183368566	$4.05114E - 03$
2.6	0.7222222222	0.7260520712	$3.82985E - 03$
2.7	0.7297297297	0.7330732332	$3.34350E - 03$
2.8	0.7368421053	0.7394770103	$2.63490E - 03$
2.9	0.7435897436	0.7453400700	$1.75033E - 03$
3.0	0.7500000000	0.7507390792	$7.39079E - 04$
3.1	0.7560975610	0.7557507047	$3.46856E - 04$
3.2	0.7619047619	0.7604516132	$1.45315E - 03$
3.3	0.7674418605	0.7649184721	$2.52339E - 03$
3.4	0.7727272727	0.7692279476	$3.49933E - 03$
3.5	0.7777777778	0.7734567083	$4.32107E - 03$
3.6	0.7826086957	0.7776814195	$4.92728E - 03$
3.7	0.7872340426	0.7819787491	$5.25529E - 03$
3.8	0.7916666667	0.7864253656	$5.24130E - 03$
3.9	0.7959183673	0.7910979325	$4.82043E - 03$
4.0	0.8000000000	0.7960731198	$3.92688E - 03$

polynomials over the interval $[1, 2]$, i.e., $S_i^{(1,2)}(x)$ to solve the problem. Let us choose $N = 4$ to approximate $f(x) \approx \sum_{i=1}^4 c_i S_i^{(1,2)}(x)$.

The proposed method gives the general shifted Genocchi coefficients as $c_1 = 23/2, c_2 = 9/2, c_3 = 2/3, c_4 = 0$, and therefore, the approximate solution produced is $f^*(x) = c_1 S_1^{(1,2)}(x) + c_2 S_2^{(1,2)}(x) + c_3 S_3^{(1,2)}(x) + c_4 S_4^{(1,2)}(x) = 2x^2 + 3x + 2$ which reproduces the exact solution over the interval $[1, 2]$. Figure 3 shows both $f(x)$ and $f^*(x)$ over the interval $[1, 2]$. The numerical results and absolute errors are shown in Table 1.

Example 2. Consider the following FIDE:

$$\begin{aligned}
 {}_0D_x^{1/2}f(x) &= \frac{\sqrt{x}(x+1)^{3/2} + (x+1) \tan h^{-1}\left(\sqrt{x/(x+1)}\right)}{(x+1)^{5/2} \sqrt{\pi}} \\
 &+ x \left(\ln\left(\frac{3}{5}\right) - 4 \right) + \int_2^4 xtf(t)dt.
 \end{aligned}
 \tag{57}$$

Let $f(2) = 2/3$, the exact solution is $f(x) = x/(x + 1)$. Let us choose orders of $N = 4$ of the general shifted Genocchi

polynomials over the interval $[a, b] = [2, 4]$ to approximate $f(x)$ and compare the graphs and absolute errors between the exact solution and approximate solutions of orders $N = 4$. Figure 4 shows both $f(x)$ and $f^*(x)$ of orders $N = 4$ over the interval $[2, 4]$. The numerical results and absolute errors are shown in Table 2, which shows our method of high accuracy.

6. Conclusion

In this paper, the 2nd-kind nonhomogeneous Fredholm FIDE is solved using the general shifted Genocchi polynomials $S_n^{(a,b)}(x)$. We introduce the general shifted Genocchi polynomials and derive the formula for computing the general shifted Genocchi coefficients. Some new properties for the general shifted Genocchi polynomials were introduced including the determinant form. Also, we derive the analytical expression of the integral of the product of the general shifted Genocchi polynomials, $T_{(n,m)}$; the integral kernel matrix, K_S ; and the general shifted Genocchi polynomial operational matrix of Caputo’s fractional derivatives, ${}_0P_S^\alpha$. By approximating each term in the FIDE in terms of the general shifted Genocchi polynomials, the equation is transformed into a system of algebraic equations. With the use of the collocation method over the interval $[a, b]$ and the initial condition given, the arbitrary domain Fredholm FIDE can be solved with very high accuracy with only few terms of the general shifted Genocchi polynomials. For future work, we hope we can extend this approach to other types of Appell polynomials, such as the Bernoulli polynomials which had been used widely such as in [27]. Apart from that, we hope we can use this general shifted Genocchi polynomial approach to solve other kinds of fractional calculus problems, such as those in [28, 29], or inverse fractional calculus problems [30, 31].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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