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On The Spectrum of the Matrix Operator $A = (a_{nk})$ on Hahn Sequence Space h

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Abstract: In this paper, first we define a matrix operator $A = (a_{nk})$ by

$$a_{n,k} = \begin{cases} \frac{1}{2} & (k = n, n-1)\ (n = 0, 1, 2....) \\ 0 & otherwise, \end{cases}$$

and we show that $A = (a_{nk})$ is a linear and bounded operator on Hahn sequence space h. Then we calculate the fine spectrum of matrix operator $A = (a_{nk})$ on the Hahn sequence space h. We also determine the point spectrum, the residual spectrum and continuous spectrum of matrix operator $A = (a_{nk})$ on Hahn sequence space h.

Keywords: The Hahn sequence space, matrix operators, spectrum of an operator.

1. Preliminaries, Background and Notation

We denote the space of all complex valued sequences by ω. Each vector subspace of ω is called as a *sequence space*, as well. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. By ϕ, we mean the space of all finitely non-zero sequences. A sequence space μ is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : \mu \to \mathbb{C}$ with $p_n(x) = x_n$ for all $x = (x_n) \in \mu$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \dots\}$. If λ is an FK-space, $\phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have AK property, where e^k is a sequence whose only term in k^{th} place is 1 the others are zero for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$. If ϕ is dense in λ, then λ is called AD-space, thus AK implies AD.
A normed FK-spaces is called a BK-space, that is, a BK-space is a Banach space with continuous coordinates, (Choudhary & Nanda, 1989, pp. 272-273). The sequence spaces ℓ_∞, c and c_0 are BK-spaces with the usual sup-norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. By ℓ_1, ℓ_p, cs, cs_0 and bs, we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where $1 < p < \infty$. Moreover, the space of all bounded variation sequences bv and the absolutely summable sequence space σ_∞ are defined, respectively by

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$$\begin{aligned}
bv &= \left\{x=(x_k)\in\omega : \sum_{k=1}^{\infty}|x_k-x_{k-1}|<\infty\right\},\\
\sigma_\infty &= \left\{x=(x_k)\in\omega : \sum_{k=1}^{\infty}\frac{1}{k}|x_k|<\infty\right\}
\end{aligned}$$

The alpha-dual λ^α, beta-dual λ^β and gamma-dual λ^γ of a sequence space λ are defined by

$$\begin{aligned}
\lambda^\alpha &:= \{x=(x_k)\in\omega : xy=(x_ky_k)\in\ell_1 \text{ for all } y=(y_k)\in\lambda\},\\
\lambda^\beta &:= \{x=(x_k)\in\omega : xy=(x_ky_k)\in cs \text{ for all } y=(y_k)\in\lambda\},\\
\lambda^\gamma &:= \{x=(x_k)\in\omega : xy=(x_ky_k)\in bs \text{ for all } y=(y_k)\in\lambda\}.
\end{aligned}$$

Let λ, μ be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k,n\in\mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A:\lambda\to\mu$, if for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n=\sum_k a_{nk}x_k \tag{1}$$

provided the series on the right side of (1) converges for each $n\in\mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda:\mu)$, we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda:\mu)$ if and only if Ax exists, i.e. $A_n\in\lambda^\beta$ for all $n\in\mathbb{N}$ and belongs to μ for all $x\in\lambda$, where A_n denotes the sequence in the n-th row of A.

2. Investigation on Hahn Sequence Space

Hahn sequence space is defined by Hahn Hahn (1922) as $BK-$space h of all sequences $x=(x_k)$ as follow

$$h=\left\{x=(x_k)\in\omega : \sum_{k=1}^{\infty}k|\Delta x_k|<\infty, \ \ and \ \ \lim_{k\to\infty}x_k=0\right\}, \tag{2}$$

where Δ denotes the forward difference operator, that is, $\Delta x_k=x_k-x_{k+1}$, for all $k\in\mathbb{N}$. Hahn Hahn (1922) proved that the sequence space h is a $BK-$space with the following norm

$$\|x\|_h=\sum_k k|\Delta x_k|+\sup_k|x_k| \tag{3}$$

which was defined by Hahn Hahn (1922) and by Goes Goes and Goes (1970). Rao (Rao, 1990, Proposition 2.1) defined a new norm on Hahn sequence space h as $\|x\|=\sum_k k|\Delta x_k|$
Hahn has also proved the followings;

$$h\subset\ell_1\cap\int c_0, \ \ where \ \int\lambda=\{x=(x_k)\in\omega : (kx_k)\in\lambda\}, \tag{4}$$

$$h^\beta=\sigma_\infty. \tag{5}$$

Goes (Goes & Goes, 1970, heorem 3.5) proved that the space h is a $BK-$space with AK. Moreover he proved that $h=\ell_1\cap\int bv=\ell_1\cap\int bv_0$ and $h=(\sigma_\infty)^\beta$ (see (Goes & Goes, 1970, heorem 3.2 and 3.4)). Then, some topological properties and results has been proved by distinguished mathematicians (see Rao (1990); Rao and Subramanian (2002a, 2002b)). Especially, Rao Rao (1990) characterized some matrix classes from Hahn sequence space h into some classical sequence spaces c, c_0, ℓ_∞, ℓ_1, and he characterized the matrix transformation from h into itself.

Theorem 0.1. *Rao (1990) $A \in (h : c_0)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = 0, \quad (k = 1,2,...) \tag{6}$$

$$\sup_{n,k} \frac{1}{k}\sum_{v=1}^{k} a_{nv} < \infty \tag{7}$$

Theorem 0.2. *Rao (1990) $A \in (h : c)$ if and only if (7) holds and*

$$\lim_{n\to\infty} a_{nk} \quad exists, \quad (k = 1,2,...). \tag{8}$$

Theorem 0.3. *Rao (1990) $A \in (h : \ell_\infty)$ if and only if (7) holds.*

Theorem 0.4. *Rao (1990) $A \in (h : \ell_1)$ if and only if*

$$\sum_{n=1}^{\infty} |a_{nk}| \quad converges, \quad (k = 1,2,...) \tag{9}$$

$$\sup_m \frac{1}{m}\sum_{n=1}^{\infty}\left|\sum_{k=1}^{m} a_{nk}\right| < \infty \tag{10}$$

Theorem 0.5. *Rao (1990) $A \in (h : h)$ if and only if (6) holds and*

$$\sum_{n=1}^{\infty} n|a_{nk} - a_{n+1,k}| \quad converges, \quad (k = 1,2,...) \tag{11}$$

$$\sup_m \frac{1}{m}\sum_{n=1}^{\infty} n\left|\sum_{k=1}^{m}(a_{nk} - a_{n+1,k})\right| < \infty \tag{12}$$

Remark 0.1. *It was shown in* (Malkowsky, Rakočević, & Tuğ, 2021, Remark 3.10) *that the condition in* (11) *is redundant, so $A \in (h,h)$ if and only if the conditions in* (6) *and* (12) *are satisfied.*

3. Spectrum of Bounded Operators

Let X be a linear space. The set of all linear operators $T : X \to X$ is denoted by $L(X)$. By $B(X)$, we denote the set of all bounded linear operators on X into itself. Suppose that X is a Banach space. Then, if $T \in B(X)$, the adjoint operator T^* of T is in $B(X^*)$, where X^* is dual space of X, defined by

$$(T^*y^*)(x) = y^*(Tx) \ \ for\, all \ \ y^* \in X^* \ \ and \ \ x \in X. \tag{13}$$

Let $T : D(T) \to X$ be a linear operator on its domain $D(T) \subset X$ into the complex normed space X. For $T \in B(X)$ we associate a complex number λ with the operator $(T - \lambda I)$ denoted by T_λ which is called the perturbed operator on the same domain $D(T)$ as T where I denotes the identity operator. The inverse $(T - \lambda I)^{-1}$, which is denoted by T_λ^{-1} is called the resolvent operator of T_λ.
A regular value of β is a complex number λ of T such that

(R_1) T_λ^{-1} exists,

(R_2) T_λ^{-1} is bounded and

(R_3) T_λ^{-1} is defined on a set which is dense in X.

Definition 0.6. *The resolvent set of T is the set of such regular values β of T and it is denoted by $\rho(T)$, that is,*

$$\rho(T) = \{\lambda \in \mathbb{C} : \alpha(T - \lambda I) = 0, \ \ R(T - \lambda I) = X\}. \tag{14}$$

Definition 0.7. *The spectrum of T denoted by $\sigma(T)$ consist of all scalars which are not in $\rho(T)$, that is*

$$\sigma(T) = \mathbb{C} \setminus \rho(T). \tag{15}$$

Thus the spectrum $\sigma(T)$ consist of those values of $\lambda \in \mathbb{C}$, for which T_λ is not invertible.
It is possible to define the spectrum $\sigma(T)$ as partitioned into three disjoint sets as follows:

(i) *$\sigma_p(T,X)$(Point-Discrete spectrum) is the set such that T_λ^{-1} does not exist. A $\lambda \in \sigma_p(T)$ is said to be an eigenvalue of T.*

(ii) *$\sigma_c(T,X)$(Continuous spectrum) is the set such that T_λ^{-1} exists and satisfies (R_3) but not (R_2) that is T_λ^{-1} is unbounded and its domain is dense in X.*

(iii) *$\sigma_r(T,X)$(Residual spectrum) is the set such that T_λ^{-1} may be bounded or not but exists and does not satisfies (R_3), that is, the domain of T_λ^{-1} is not dense in X.*

Since these subsets are three subdivision disjoint sets, we have

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_r(T,X) \cup \sigma_c(T,X). \tag{16}$$

It is well know that some of these sets defined above can be empty. The problem that we have to discuss is an existence problem. If X is finite-dimensional space, then the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$, that is, $\sigma_r(T,X) = \sigma_c(T,X) = \emptyset$. Moreover, only the point spectrum $\sigma_p(T,X)$ can be empty for some operator on sequence spaces. Thus, we should consider to prove all sub-spectrum of an operator in the infinite dimensional case.
Now the definition of three more subdivision of spectrum of an operator named as approximate spectrum, defect spectrum and compression spectrum which were defined by Appell at al Appell, De Pascale, and Vignoli (2008).

Definition 0.8. *Appell et al. (2008) Let X is a Banach space and $T \in B(X)$. An $x = (x_k)$ Weyl sequence for T defined by $\|x\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$. Then the approximate spectrum, defect spectrum and compression spectrum are defined as follow*

(i) *The approximate spectrum is the set of $\lambda \in \mathbb{C}$ such that there exists a Weyl sequence for $\lambda I - T$, that is,*

$$\sigma_{ap}(T,X) = \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - T\} \tag{17}$$

(ii) *The defect spectrum of T is the set of $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not surjective, thet is,*

$$\sigma_\delta(T,X) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective } \} \tag{18}$$

(iii) *The compression spectrum is the set of $\lambda \in \mathbb{C}$ such that the range of $\lambda I - T$ id not dense in X, that is,*

$$\sigma_{co}(T,X) = \left\{\lambda \in \mathbb{C} : \overline{R(\lambda I - T)} \neq X\right\}. \tag{19}$$

The first two spectrum defined as (0.8) and (0.8) which are not needed to be disjoint gives the following

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X), \tag{20}$$

and the compression spectrum gives another(not necessarily disjoint) decomposition of the spectrum as

$$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X). \tag{21}$$

Proposition 0.9. *Let X is a Banach space and $T \in B(X)$. Then the followings hold.*

(i) $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$,

(ii) $\sigma_{ap}(T,X) \subseteq \sigma_{\delta}(T,X)$,

(iii) $\sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_p(T,X)$,

(iv) $\sigma_c(T,X) = \sigma(T,X) \setminus [\sigma_p(T,X) \cup \sigma_{co}(T,X)]$

Sometimes it is useful to connect the spectrum of a bounded linear operator with its adjoint and to set up existence and uniqueness results for linear operator equations in Banach spaces and their adjoints. The following proposition gives this relation.

Proposition 0.10. *(Appell et al., 2008, Proposition 1.3, p. 28) The following relations on the spectrum and subspectrum of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ hold:*

(i) $\sigma(T^*,X^*) = \sigma(T,X)$.

(ii) $\sigma_c(T^*,X^*) \subseteq \sigma_{ap}(T,X)$.

(iii) $\sigma_{ap}(T^*,X^*) = \sigma_{\delta}(T,X)$.

(iv) $\sigma_{\delta}(T^*,X^*) = \sigma_{ap}(T,X)$.

(v) $\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$.

(vi) $\sigma_{co}(T^*,X^*) \supseteq \sigma_p(T,X)$.

(vii) $\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

4. The Spectrum of the Matrix Operator $A = (a_{nk})$ on Hahn Sequence Space h

Operator theory is one of the part of Functional analysis where significant applications have been done in several sciences. One of the mathematical application is spectral theory. Literature includes many significant works on spectrum of bounded operators on sequence spaces.
The Hahn sequence space defined by Hahn Hahn (1922) and studied by many mathematician (see Rao (1990); Rao and Subramanian (2002a, 2002b)). Kirišci Kirişci (2013); Kirisci (2013); Kirişci (2014) compiled all works on Hahn sequence space in his remarkable papers. In Kirişci (2013), he defined a new Hahn sequence space derived by Cesàro Mean. Moreover, in Kirişci (2014) he defined $p-$Hahn sequence space. Yeşilkayagil and Kirişci Yeşilkayagil and Kirişci (2014) studied the fine spectrum of forward difference operator Δ on the Hahn Space h.
Most recently Malkowsky et al Malkowsky, Rakočević, and Tuğ (2021) and Malkowsky Malkowsky (2021) studied the generalized Hahn sequence space h_d, where $d = (k_k)$ is monotonically increasing sequence, and showed some compact operators on the new Hahn sequence space. Moreover, Malkowsky et al Malkowsky, Milovanović, Rakočević, and Tuğ (2021) and Tuğ et al Tuğ, Rakočević, and Malkowsky (2021) studied the spectrum of the generalized difference operator Δ_i^3 of order three operator on Hahn sequence space h and the generalized difference operator Δ_i^3 domain on Hahn sequence space h and some matrix transformations from and into the space h, respectively.
In this section, we define the matrix operator A and then we show that this operator is a bounded operator on h. Then we calculate the spectrum and subdivision of spectrum of the operator A on Hahn sequence space h.
Let the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ is defined as $A_n x = \frac{1}{2}(x_n + x_{n-1})$ for all $n = \{0,1,2,...\}$, that is,

$$a_{n,k} = \begin{cases} \frac{1}{2} & (k = n, n-1)\ (n = 0,1,2....) \\ 0 & (otherwise), \end{cases}$$

Then the following holds:

Proposition 0.11. *We have $A \in (h,h)$ and*

$$\|L_A\|_{(h,h)} = 1, \quad where \;\; L_A(x) = A_x \; for \; all \; x \in h. \tag{22}$$

Proof. We write

$$h(z) = \sum_{k=1}^{\infty} k|z_k - z_{k+1}| \; for \; all \; z \in (k^{-1} * \ell_1)$$

and

$$\|z\|_\infty = \sup_{k\geq 0} |z_k| \; for \; all \; z \in c_0$$

hence

$$\|z\|_h = h(z) + \|z\|_\infty \; for \; all \; z \in h$$

By (Goes & Goes, 1970, Theorem 3.5), h is a BK space with AK with respect to $\|x\|_h (x \in h)$. Thus every $B \in (h,h)$ defines an operator $L_B \in B(h,h)$, where $L_B(x) = B_x \;\; for \; all \;\; x \in h$ (Wilansky, 2000, Theorem 4.2.8) and conversely, every operator $L \in B(h,h)$ is given by a matrix $B \in (h,h)$, where $B_x = L(x)$ for every $x \in h$ (Jarrah & Malkowsky, 2003, Theorem 1.9).
We write $y = Ax \; for \; allx \in h$. Then obviously $y \in c_0$ and $\|y\|_\infty \leq \|x\|_\infty$, and

$$\begin{aligned} h(y) = \sum_{k=1}^{\infty} k|\Delta y_k| &= \sum_{k=1}^{\infty} k|A_k x - A_{k+1}x| = \frac{1}{2}\sum_{k=1}^{\infty} k|x_k + x_{k+1} - (x_{k+1} + x_{k+2})| \\ &= \frac{1}{2}\sum_{k=1}^{\infty} k|x_k - x_{k+2}| \leq \frac{1}{2}\left(h(x) + \sum_{k=2}^{\infty}(k-1)|x_k - x_{k+1}|\right) \\ &\leq h(x) \end{aligned}$$

and so $L_A \in B(h,h)$ with

$$\|L_B\|_{h,h} \leq 1. \tag{23}$$

Let $\varepsilon > 0$ be given. We choose $m \in \mathbb{N}$ such that $m > \frac{1}{(2\varepsilon)-1}$. We put $x = e^{[m]} = \sum_{k=0}^{m} e^{(k)}$. Then obviously $x \in h$ and we obtain

$$\begin{aligned} \|x\|_h &= \sum_{k=1}^{\infty} k|x_k - x_{k+1}| + 1 = m + 1, \\ h(y) &= \frac{1}{2}\sum_{k=1}^{\infty} k|x_k - x_{k+2}| = \frac{1}{2}(m - 1 + m), \end{aligned}$$

hence

$$\|y\|_{(h,h)} = m + \frac{1}{2}.$$

Then we have

$$\frac{\|y\|_h}{\|x\|_h} = \frac{m + \frac{1}{2}}{m+1} = 1 - \frac{1}{2(m+1)} > 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies

$$\|y\|_h \geq \|x\|_h,$$

and then

$$\|L_A\|_{(h,h)} \geq 1 \tag{24}$$

So (24) and (23) together imply (22) as proposed in the theorem. □

Theorem 0.12. $\sigma(A,h) = \left\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$

Proof. Suppose that $|\lambda - \frac{1}{2}| > \frac{1}{2}$. Since the matrix A is triangular, then $(A-\lambda I)^{-1} = B = (b_{nk})$ is given by

$$(b_{nk}) = \begin{bmatrix} \frac{1}{2}-\lambda & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2}-\lambda & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2}-\lambda & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{1-2\lambda} & 0 & 0 & \cdots \\ \frac{-2}{(1-2\lambda)^2} & \frac{2}{1-2\lambda} & 0 & \cdots \\ \frac{2}{(1-2\lambda)^3} & \frac{-2}{(1-2\lambda)^2} & \frac{2}{1-2\lambda} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

where

$$b_{nk} = \begin{cases} \frac{2}{1-2\lambda} & , \quad k=n \\ (-1)^{n-k}\frac{2}{(1-2\lambda)^{n-k+1}} & , \quad k \leq n-1 \\ 0 & , \quad otherwise \end{cases}$$

The aim of us here is to show that the matrix B is a bounded linear operator on h. To see this, we apply the Theorem 0.5 and Remark 0.1. Thus, we should show that the following conditions

$$\begin{aligned} &(i) \quad \lim_{n\to\infty} b_{nk} = 0, \ \ for \ \ each \ \ k = 1,2,... \\ &(ii) \quad \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\sum_{k=1}^{m}(b_{nk} - b_{n+1,k})\right| < \infty \end{aligned}$$

are satisfied for the matrix B.
Since we have

$$b_{nk} = \frac{2}{1-2\lambda}\left(\frac{-1}{1-2\lambda}\right)^{n-k}. \tag{25}$$

for each $k \in \mathbb{N}$, and $|\lambda - \frac{1}{2}| > \frac{1}{2}$, then the expression $\left|\frac{-1}{1-2\lambda}\right| < 1$. Therefore, we can easily see that $\lim_{n\to\infty} b_{nk} = 0$ for each k. It proves that (i) is satisfied.
To see the condition (ii) is also satisfied we have

$$\begin{aligned} & \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\sum_{k=1}^{m}(b_{nk} - b_{n+1,k})\right| \\ = \; & \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\sum_{k=1}^{m}\left(\frac{2}{1-2\lambda}\left(\frac{-1}{1-2\lambda}\right)^{n-k} - \frac{2}{1-2\lambda}\left(\frac{-1}{1-2\lambda}\right)^{n-k+1}\right)\right| \\ = \; & \left|\frac{4-4\lambda}{(1-2\lambda)^2}\right| \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\sum_{k=1}^{m}\left(\frac{-1}{1-2\lambda}\right)^{n-k}\right| \\ = \; & \left|\frac{2-2\lambda}{\lambda(1-2\lambda)^2}\right| \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\left(\frac{-1}{1-2\lambda}\right)^{n} - \left(\frac{-1}{1-2\lambda}\right)^{n-m-1}\right| \\ \leq \; & \left|\frac{2-2\lambda}{\lambda(1-2\lambda)^2}\right| \left(\sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\left(\frac{-1}{1-2\lambda}\right)^{n}\right| + \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\left(\frac{-1}{1-2\lambda}\right)^{n-m-1}\right|\right) \end{aligned}$$

Since $|\lambda - \frac{1}{2}| > \frac{1}{2}$ and by using the ratio test we can say that

$$\|B\|_{(h;h)} = \sup_m \frac{1}{m}\sum_{n=1}^{\infty} n \left|\sum_{k=1}^{m}(b_{nk} - b_{n+1,k})\right| < \infty.$$

Now suppose that $\lambda \in \mathbb{C}$ which satisfies $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$. If $\lambda \neq \frac{1}{2}$, then $A - \lambda I$ is triangular and has the inverse $(A - \lambda I)^{-1}$. Let decide $y = (1, 0, 0, 0, \cdots)$ which is in h, then $(A - \lambda I)^{-1} y = x$ gives

$$x_n = \frac{2}{1 - 2\lambda} \left(\frac{-1}{1 - 2\lambda} \right)^n. \tag{26}$$

Since $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$, the sequence x_n does not convergence to 0 and we may say that $x = (x_n) \notin h$. Thus the inverse $(A - \lambda I)^{-1}$ is not a map in the class $(h : h)$ and so $\lambda \in \sigma(A, h)$.
If $\lambda = \frac{1}{2}$, then the operator $A - \lambda I$ which is represented as

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Since the range of $A - \lambda I$ is not dense in h, then $\lambda \in \sigma(A, h)$ which gives us the result

$$\left\{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2} \right\} \subseteq \sigma(A, h).$$

□

Theorem 0.13. $\sigma_p(A, h) = \emptyset$

Proof. Suppose that $Ax = \lambda x$ for all corresponding eigenvectors $x \neq 0$ of eigenvalues $\lambda \in \mathbb{C}$ in Hahn sequence space h. When we solve the system of linear equations

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \lambda . \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

we have the following iteration

$$\begin{aligned} \frac{1}{2}x_0 &= \lambda x_0 \implies \lambda = \frac{1}{2} \\ \frac{1}{2}x_0 + \frac{1}{2}x_1 &= \lambda x_1 \implies \frac{1}{2}x_0 + \frac{1}{2}x_1 = \frac{1}{2}x_1 \implies \frac{1}{2}x_0 = 0 \implies x_0 = 0 \\ \frac{1}{2}x_1 + \frac{1}{2}.x_2 &= \lambda x_2 \implies \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}x_2 \implies \frac{1}{2}x_1 = 0 \implies x_1 = 0 \\ &\vdots \\ \frac{1}{2}x_{n-1} + \frac{1}{2}x_n &= \lambda x_n \implies x_n = 0 \\ &\vdots \end{aligned}$$

We have $x_k = 0 \ \ \forall k \in \mathbb{N}$ which is a contradict. Moreover, if the first non zero eigenvector is x_1, then we see by the above system of equations that $x_1 = x_2 = ... = 0$. Therefore, the set of all point spectrum of the matrix A is empty set. □

If $T \in B(h)$ with a matrix A, then the adjoint operator $T^* : h^* \to h^*$ is defined by the transpose A^t of the matrix A. So we should note here that the dual space h^* of h is isometrically isomorphic to the Banach space σ_∞ of absolutely summable sequences which is normed by $\|x\| = \sup_n \frac{1}{n} |\sum_{k=1}^{n} x_k|$.

Theorem 0.14. $\sigma_p(A^*,h^*) = \left\{x \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\right\}$

Proof. Suppose that $A^*x = \lambda x,\ s \in \rho_\infty, x \neq 0$. Since A is a triangular real valued matrix, so

$$A^* = A^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$A^*x = \lambda x,\ \forall\lambda \in \mathbb{C},\ \forall x_k \neq 0,\ \forall k \in \mathbb{N} \implies \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = \lambda \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

gives us the following system and we have the following generalization of x_n as

$$\begin{aligned} \frac{1}{2}x_0 + \frac{1}{2}x_1 &= \lambda x_0 \implies \frac{1}{2}x_1 = (\lambda - \frac{1}{2})x_0 \implies x_1 = 2(\lambda - \frac{1}{2})x_0 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 &= \lambda x_1 \implies \frac{1}{2}x_2 = (\lambda - \frac{1}{2})x_1 \implies x_2 = 2(\lambda - \frac{1}{2})x_1 = 2^2(\lambda - \frac{1}{2})^2 x_0 \\ &\vdots \\ \frac{1}{2}x_{n-1} + \frac{1}{2}x_n &= \lambda x_{n-1} \implies \frac{1}{2}x_n = (\lambda - \frac{1}{2})x_{n-1} \implies x_n = 2^n(\lambda - \frac{1}{2})^n x_0. \\ &\vdots \end{aligned}$$

Now we should show that $x_n \in h^* = \sigma_\infty = \left\{x_k : \sup_n \frac{1}{n}|\sum_{k=1}^n x_k| < \infty\right\}$. For this

$$\begin{aligned} \sup_n \frac{1}{n}\left|\sum_{k=1}^n 2^k \left(\lambda - \frac{1}{2}\right)^k x_0\right| &\leq \sup_n \frac{1}{n}\sum_{k=1}^n \left|2^k \left(\lambda - \frac{1}{2}\right)^k\right| |x_0| \\ &\leq |x_0| \sup_n \frac{1}{n}\sum_{k=1}^n |2^k| \left|\lambda - \frac{1}{2}\right|^k \\ &\leq |x_0| \sup_n \frac{1}{n}\sum_{k=1}^n 2^k \frac{1}{2^k} \\ &\leq |x_0| < \infty \end{aligned}$$

Which gives us the expected result. □

Lemma 0.15. *(Goldberg, 2006, p. 59) An operator T has dense range if and only if T^* is one to one.*

Lemma 0.16. *(Goldberg, 2006, p. 60) The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

Theorem 0.17. $\sigma_r(A,h) = \sigma_p(A^*,h^*)$

Proof. If $\lambda \neq \frac{1}{2}$, then the operator $A - \lambda I$ is triangular matrix and has an inverse, that is, $(A - \lambda I)^{-1}$ exists. If $\lambda = \frac{1}{2}$ then $x_1 = x_2 = \ldots = x_k = 0$ which says that $x = 0$. It can be said from the above argument that $A - \lambda I$ has an inverse for $\lambda \in \sigma_p(A^*,h^*)$. Since $A^* - \lambda I$ is not one-to-one by Theorem (0.14), Lemma (0.15) says the result that the range $A - \lambda I$ is not dense in h. This concludes the proof. □

Theorem 0.18. $\sigma_c(A,h) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| = \frac{1}{2}\right\}$

Proof. Suppose that $\lambda \in \mathbb{C}$ such that $\left|\lambda - \frac{1}{2}\right| = \frac{1}{2}$. Since $\lambda \neq \frac{1}{2}$, $A - \lambda I$ is triangular and hence has an inverse. Consider the adjoint operator $A^* - \lambda I$. Then the linear system of homogeneous equations in the matrix form $(A^* - \lambda I)x = 0$ gives us that

$$x_n = 2^n \left(\lambda - \frac{1}{2}\right)^n x_0, \; for \; n \geq 1. \tag{27}$$

Since $\lambda \neq \frac{1}{2}$, we have $x = (x_1, x_2, ..., x_n, ...) \in h$ if and only if $x = 0$. So, $A^* - \lambda I$ is one-to-one. By Lemma (0.15) we can say that the range of $A - \lambda I$ is dense in h. This is what we wished to prove. □

Corollary 0.19. *The following consequences hold:*

(i) $\sigma_{ap}(A,h) = \sigma(A,h)$.

(ii) $\sigma_{\delta}(A,h) = \sigma(A,h)$.

(iii) $\sigma_{co}(A,h) = \{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\}$.

5. Conclusion

Spectral theory of linear and bounded operators is one of the aplied areas of the matrix and summability theory. Many significant papers have been delivered on spectrum of several linear and bounded operators on several sequence spaces. The fine spectrum and its subdivision of some linear and bounded operators on Hahn sequence space h have been calculated by some distinguised mathematicians (see Das (2017); Durna (2020); El-Shabrawy and Abu-Janah (2018); Malkowsky, Milovanović, et al. (2021); Yeşilkayagil and Kirişci (2014)).
In this paper, we defined the matrix operator $A = (a_{nk})$ and calculated the fine spectrum and its some subdivisions of the matrix operator A on the Hahn sequence space h. The results that we optained is an application of the paper studied by Das Das (2017).
As a natural continuation of this paper, the fine spectrum and its subdivisions of the matrix operator

$$a_{n,k} = \begin{cases} \frac{1}{3} & (k = n, n-1, n-2) \; (n = 0, 1, 2....) \\ 0 & otherwise, \end{cases}$$

can be calculated on Hahn sequence space h. Moreover, the operator can be generalized to the operator $A^{\ell} = (a^{\ell}_{nk})$, where $A^{\ell}_n x = \frac{1}{\ell+1} \sum_{k=0}^{l} x_{n-k}$ for all $n = 0, 1, 2, 3, ...$ and for every $x \in \omega$. Then the fine spectrum of the operator $A^{\ell} = (a^{\ell}_{nk})$ can be calculated on the sequence space h or generalized Hahn sequence space h_d which defined by Malkowsky, Rakočević, and Tuğ (2021) and recently studeied by several mathematicians (see Dolićanin-Đekić and Gilić (2022); Veličković, Malkowsky, and Dolićanin (2022); Yaying, Kirişçi, Hazarika, and Tuğ (2022)).

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