B-Spline Cubic Finite Element Method for Solving Ordinary Differential Equations

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Abstract: This work proposes numerical solution of ordinary differential equations. The proposed method is based on applying modified cubic B-spline finite element method. The existence and uniqueness for the variational form proved. The convergence of the presented scheme is given. Numerical experiments are considered to confirm our theoretical results.

1. Introduction

In the last century, the Finite Element and Finite difference methods have been one of the most successful and numerical methods for solving ordinary and partially differentials equations (Sabawi, 2017; Cangiani, Georgoulis, & Sabawi, 2018; Cangiani, Georgouils, & Sabawi, 2020; Sabawi, 2021; Sabawii, 2019; Sabawi, 2020; Sabawi, 2021; Hussein, 2011; Manaa, Moheemmeed, & Hussien, 2010) Finite Element Methods with modified B-Splines describes new weighted approximation techniques, which combine the computing advantages of B-splines and standard finite elements (Höllig, 2003). The cubic B spline-finite element method is a well-known numerical method for the numerical solution of differential equations.

Many people have used cubic B-spline to approximate numerical solution of ordinary differential equation. For example, B-spline method for solving differential equation (Sabawi, Pirdawood, & Sadeeq, 2021), quasi wavelet methods (Wenting Long, 2012) , colocation method for solving boundary value problem (Al-Humedi & jameel, 2020) , cubic B spline finite element method in polar coordinates (Li Ronglin, 1998) , Munguia and Bhatta in (Munguia & Bhatta, 2015)discussed, finite element method (Gockenbach, 2006; Jinming Wu, 2011), Galerkin methods (Sharma, Arora, & Kumar, 2013) , finite deference method (Ahmed, 2017; KrystynaStys, 2000; Pirdawood & Sabawi, 2021), B-spline method (Jameel, 2020; Siddiqi, Shahid, Arshed, & Saima, 2014; Nielsen, 1998).

The main goal of the paper is to use modified cubic B-spline finite element method for solving boundary value problem.

This work is organised as follows. In Section 2, model problem with weak form are introduced. In Section 3, finite element approximation with modified cubic B-spline are given. Error analysis is proved in section 4, finally, numerical experiments with different examples are shown in section 5.

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2. Model Problem

Consider the boundary value problem

$$-z''(s) + b(s)z'(s) + c(s)z(s) = f(s)\,, \tag{1}$$

With boundary conditions

$$z(a) = z(b) = 0 \ \ , s \in (0,1),$$

Where $b(s)$, $c(s)$ and $f(s)$ are continuous real-valued functions on the interval $[0, 1]$.

To define weak form for (1), we multiply (1) by a test function $v \in H_0^1(0,1)$, gives

$$\int_0^1 -z''(s)v(s)ds + \int_0^1 b(s)z'(s)v(s)ds + \int_0^1 c(s)z(s)v(s)ds = \int_0^1 f(s)v(s)ds. \tag{2}$$

Integrating the first term on the left-hand side of above equation by parts, imply that

$$\int_0^1 -z''(s)v(s)ds = z'(s)v(s)|_0^1 - \int_0^1 z'(s)v'(s)ds.$$

Applying boundary conditions in above, becomes

$$\int_0^1 z'(s)v'(s)ds + \int_0^1 b(s)z'(s)v(s)ds + \int_0^1 c(s)z(s)v(s)ds = \int_0^1 f(s)v(s)ds. \tag{3}$$

Above equation can be translated to bilinear form such that

$$a(z,v) = F(v) \quad for\ all\ \ z, v\, \epsilon H_0^1(0,1)\,, \tag{4}$$

Where

$$a(z,v) = \int_0^1 z'(s)v'(s)ds + \int_0^1 b(s)z'(s)v(s)ds + \int_0^1 c(s)z(s)v(s)ds$$

$$F(v) = \int_0^1 f(s)v(s)ds.$$

Lemma 2.1 (Existence, uniqueness)

Let $a(z,v)$ be bilinear form defined $c(s) \le M_2$ and $P_1 \le b(s) \le P_2$, then $a(z,v)$ is continuous and bounded. such that

$$a(w,w) \ge C_{covr}\ \|w\|_a^2, for\ all\ w \in H_0^1(0,1) \tag{5}$$

$$a(w,v) \le C_{cont}\|u\|_a.\|v\|_a, for\ all\ u, v\ \in H_0^1(0,1) \tag{6}$$

Proof: From (4) with applying Cauchy-Schwarz inequality, becomes

$$a(w,v) = \left| \int_0^1 w'(s)v'(s)ds + \int_0^1 b(s)w'(s)v(s)ds + \int_0^1 c(s)w(s)v(s)ds \right|$$

$$\leq max\,\{b_{max}, c_{max}\}\left(\int_0^1 |w'(s)v'(s)ds| + \int_0^1 |w'(s)v(s)ds| + \int_0^1 |w(s)v(s)ds| \right)$$

$$\leq c\{b_{max}, c_{max}\}\left\{ \left(\int_0^1 \left(w'(s)\right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 \left(v'(s)\right)^2 ds \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_0^1 \left(w'(s)\right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 \left(v(s)\right)^2 ds \right)^{\frac{1}{2}} + \left(\int_0^1 (w)^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 \left(v(s)\right)^2 ds \right)^{\frac{1}{2}} \right\}$$

$$\leq c\{b_{max}, c_{max}\}\left\{ \left(\int_0^1 \left(w'(s)\right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 \left(v'(s)\right)^2 ds \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_0^1 \left(w'(s)\right)^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 \left(v(s)\right)^2 ds \right)^{\frac{1}{2}} \right\}$$

$$+ \left(\int_0^1 (w)^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 \left(v(s)\right)^2 ds \right)^{\frac{1}{2}}$$

$$\leq max\{b_{max}, c_{max}\}\left(\|w(s)\|_a \|v(s)\|_a + \|w(s)\|_a \|v(s)\|_{L^2(0,1)} + \|w(s)\|_{L^2(0,1)} \|v(s)\|_{L^2(0,1)} \right).$$

Using Poincar´e-Friedrichs inequality $\|v(s)\|_{L^2(0,1)} \leq C\|v(s)\|_{H_0^1(0,1)} = C\|v(s)\|_a$ and setting

$C_{cont}= C_P max\{b_{max}, c_{max}\}$,

The first part of lemma will be followed, the second part of lemma so that

$$a(w,w) = \int_0^1 w'(s)w'(s)ds + \int_0^1 b(s)w'(s)w(s)ds + \int_0^1 c(s)w(s)w(s)ds.$$

The first and third term on above equation can be bounded as

$$\int_0^1 w'(s)w'(s)dx + \int_0^1 c(s)w(s)w(s)ds \geq \int_0^1 \left(w'(s)\right)^2 ds \geq \frac{1}{1+c}\|w(s)\|^2_{H_0^1(0,1)} = \frac{1}{1+c}\|w(s)\|^2_a,$$

and

$$\int_0^1 b(s)v'(s)v(s)ds = -\frac{1}{2}\int_0^1 b'(s)\left(v(s)\right)^2 ds \geq -\frac{L_2}{2}\int_0^1 \left(v(s)\right)^2 ds \geq -\frac{L_2}{2}\|w(s)\|^2_{H_0^1(0,1)}$$
$$= -\frac{L_2}{2}\|w(s)\|^2_a.$$

Collection tougher, gives

$$a(w,w) \geq \left(\frac{1}{1+c} - \frac{L_2}{2}\right)\|w(s)\|^2_a,$$

Where $C_{covr} = \left(\frac{1}{1+c} - \frac{T_2}{2} - KR\right)$, the results will be finished

3. Finite Element Approximation

Let S^h is a finite dimensional space such that $S^h \subseteq H_0^1(\Omega)$. To develop the numerical method for approximating solution of boundary value problems (1) the interval [0,1] is partitioned into N +1 uniformly spaced points x_i such that $0 < s_0 < s_1 < \cdots < s_{N-1} < s_{N=1}$ and $h = 1/(N+1)$.

Setting

$$S^h = \{v \in S_3 \in C^2[0,1]: v(0) = v(1) = 0\}, \qquad [7]$$

Where P_3 is the space of all polynomials of degree ≤ 3.

We seek to find an approximation $\mathbf{Z_h(s) \epsilon S^h}$ such that

$$\mathrm{a}(\mathrm{Z_h}, \mathrm{v_h}) = \mathrm{F}(\mathrm{v_h}), \quad \mathrm{v_h} \in \mathrm{S^h}, \qquad [8]$$

Where

$$a(Z_h, v_h) = \int_0^1 Z_h' v_h' ds + a(Z_h, v_h)\int_0^1 b(s)Z_h'(s)v_h(s)ds + \int_0^1 c(s)Z_h(s)v_h(s)ds \qquad [9]$$

$$F(v_h) = \int_0^1 f(s)v_h(s)ds.$$

The typical third-degree B-spline basis as

$$B_i^3(s) = \begin{cases} h^{-3} g_1(s - s_{i-2}), & s \in [s_{i-2}, s_{i-1}] \\ g_2\left(\frac{s - s_{i-1}}{h}\right), & s \in [s_{i-1}, s_i] \\ g_2\left(\frac{s_{i+1} - s}{h}\right), & s \in (s_i, s_{i+1}) \\ h^{-3} g_1(s_{i+2} - s), & s \in (s_{i+1}, s_{i+2}) \\ 0, & otherwise. \end{cases} \quad [10]$$

Where

$$g_1(s) = s^3 \ , g_2(s) = 1 + 3s + 3s^2 - 3s^3,$$

for $i = -1, 0, \dots, N+1.$ For a sufficiently smooth function z(s) there always exists a unique third-degree spline $Z_h(s)$ such that

$$Z_h = \sum_{m=-1}^{N+1} \alpha_i B_i(s), \quad [11]$$

where α_i are unknown quantities to be determined from (11). For the sake of simplicity, using $z_i = Z_h(s_{(i}, t)$

$$\begin{cases} z_i = \alpha_{i-1} + 4\alpha_i + \alpha_{i+1} \\ z_i' = \frac{3}{h}(\alpha_{i+1} - \alpha_{i+1}) \\ z_i'' = \frac{6}{h^2}(\alpha_{i-1} - 2\alpha_i + \alpha_{i+1}). \end{cases} \quad [12]$$

Using (12) and boundary conditions (2), gives

$$\begin{cases} a_1 = -a_{-1} - 4a_0 \\ a_{N+1} = -a_{N-1} - 4a_{N.} \end{cases} \quad [13]$$

Substituting (13) in (9), we have

$$Z_h(s) = a_0[B_0(s) - 4B_{-1}(s)] + a_1[B_1(s) - B_{-1}(s)] + \dots\dots +$$

$$a_{N-1}[B_{N-1}(s) - B_{N-1}(s)] + a_N[B_N(s) - 4B_{N+1}(s)],$$

So that

$$Z_h(s) = a_0\overline{B_0}(s) + a_1\overline{B_1}(s) + \dots\dots + a_{N-1}\overline{B_{N-1}}(s) + a_N\overline{B_N}(s),$$

Where

$$\begin{cases} \overline{B_0}(s) = B_0(s) - 4B_{-1}(s) \ , \quad \overline{B_1}(s) = B_1(s) - B_{-1}(s) \\ \bar{B}_i(s) = B_i(s) \ , i = 2,3, \dots\dots, N-2 \\ \overline{B_{N-1}}(s) = B_{N-1}(s) - B_{N-1}(s) \ , \quad \overline{B_N}(s) = B_N(s) - 4B_{N+1}(s) \end{cases} \quad [14]$$

Table 1: The values of $B_i(x)$ and their derivatives can be tabulated as in Table 1.

	x_{i-1}	x_i	x_{i+1}	Else
$B_i(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$B_i'(x)$	$-\frac{1}{2h}$	0	$\frac{1}{2h}$	0
$B_i''(x)$	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

Inserting (11) in (4) with choosing $v_h = \bar{B}_m(s)$, gives

$$\sum_{i=0}^{N} \alpha_i \left(\int_a^b (\bar{B}_i)'(s)(B_m)'(s)ds + \int_a^b (\bar{B}_i)'(s)\bar{B}_m(s)ds + \int_a^b \bar{B}_i(s)\bar{B}_m(s)ds \right) = \int_a^b f(s)\bar{B}_m(s)ds. \quad (15)$$

Where

$$A^e = \int_a^b (\bar{B}_i)'(s)(\bar{B}_m)'(s)ds = \frac{3}{10h}\begin{bmatrix} 80 & 43 & -20 & -1 \\ 43 & 104 & -14 & -24 \\ -20 & -14 & 80 & -15 \\ -1 & -24 & -15 & 80 \end{bmatrix}, \quad [16]$$

$$B^e = \int_0^1 (\bar{B}_i)'(s)\bar{B}_m(s)ds = \frac{1}{20}\begin{bmatrix} 0 & 133 & 52 & 1 \\ -133 & 0 & 244 & 56 \\ -52 & -244 & 0 & 245 \\ -1 & -56 & -245 & 0 \end{bmatrix}, \quad [17]$$

$$C^e = \int_0^1 \bar{B}_i(s)\bar{B}_m(s)ds = C = \frac{h}{140}\begin{bmatrix} 496 & 773 & 116 & 1 \\ 773 & 2296 & 1190 & 120 \\ 116 & 1190 & 2416 & 1191 \\ 1 & 120 & 1191 & 2416 \end{bmatrix}. \quad [18]$$

Finally, to compute $\int_a^b f(s)\bar{B}_m(s)ds$, applying the basic 3-point Gauss quadrature is defined by on [0.1], imply that

$$\int_{-1}^{1} f(t)dt = \sum_{j=1}^{3} C_j f(\xi_j), \quad [19]$$

where the nodes $\xi_i$and weights C_i are

$$\xi_1 = -\sqrt{\frac{3}{5}} \quad , \quad \xi_2 = 0 \quad , \quad \xi_3 = \sqrt{\frac{3}{5}} \tag{20}$$

$$C_1 = \frac{5}{9} \quad , \quad C_2 = \frac{8}{9} \quad , \quad C_3 = \frac{5}{9}, \tag{21}$$

for $g(s)$ on $[0,1]$, with substitution

$$s = s_k + \frac{h}{2}(1+t) \ , \quad t \in [-1,1] \, . \tag{22}$$

Using (19) we have

$$\int_0^1 g(s)ds = \sum_{k=0}^{N} \int_{s_k}^{s_{k+1}} g(s)ds = \frac{h}{2}\sum_{k=0}^{N} \int_{-1}^{1} g\left(s_k + \frac{h}{2}(1+t)\right)ds \tag{23}$$

$$\approx \frac{h}{2}\sum_{k=0}^{N}\sum_{j=1}^{3} C_j g(\xi_{k,j})$$

$$\xi_{k,j} = \frac{h}{2}\xi_j + \frac{s_k + s_{k+1}}{2} \quad , \quad k = 0, \ldots\ldots, N \ , \quad i = 1,2,3. \tag{24}$$

Using (24) and (20) we have

$$\xi_{k,1} = s_k + \frac{1-\sqrt{\frac{3}{5}}}{2}h \ , \ \xi_{k,2} = s_k + \frac{h}{2} \ , \ \xi_{k,3} = s_k + \frac{1+\sqrt{\frac{3}{5}}}{2}h \ , \quad k = 0, \ldots\ldots, N. \tag{25}$$

Using (23) the F_i given in (15) can be approximated by

$$F_i \approx \frac{h}{2}\sum_{k=0}^{N}\sum_{j=1}^{3} C_i f\left(\xi_{k,j}\right)\bar{B}_i\left(\xi_{k,j}\right) \ , \ i = 0, \ldots., N+1 \, . \tag{26}$$

Collecting all of these results together, we have the following system

$$(A^e + bB^e + cC^e)\alpha = F^e,$$

where $\alpha = [\alpha_{i-1}, \alpha_i, \alpha_{i+1}, \alpha_{i+2}]^T$. To find the coefficient $\alpha_0, \alpha_1, \ldots\ldots, \alpha_N$, we assemble all the matrices A^e, B^e, C^e so that above equation becomes

$$(A + bB + cC)\alpha = F,$$

where

$$A = \frac{3}{10h}(-1,-24,-15,80,-15,-24,-1), \qquad B = \frac{1}{20}(1,56,-245,0,-245,56,1),$$

$$C = \frac{h}{140}(1, 120,1191,2416,1191,120,1).$$

4. Convergence Analysis

This section aims to present the error analysis theorems for the proposed method. To start with, we introduce the energy norm $\|z\|_a$ defined by

$$\|z\|_a = \left(a(z,z)\right)^{1/2} = \left[\int_0^1 (C(s)\|z'(s)\|^2 + \|z(s)\|^2)ds\right]^{1/2} = \|z\|_{H^1(0,1)}.$$

Suppose that z is the exact solution of the problem (4) and Z_h be its approximate solution of (8), then we have

Since $v \in H_0^1(\Omega)$, setting $v = v_h \in S^h$ from (4) we have

$$a(z, v_h) = F(v_h), \qquad v_h \in S^h$$

Also, by, (8),

$$a(Z_h, v_h) = F(v_h), \qquad v_h \in S^h$$

Subtracting above equation, we deduce the Galerkin orthogonal.

$$a(z - Z_h, v_h) = 0, \forall v_h \in S^h.$$

Lemma (3.1): [Cea's lemma]. Let $z(s)$ be the solution to the Dirichlet boundary value problem (4) and $Z_h(s)$ its finite element approximation given by (8), then

$$\|z(s) - Z_h(s)\|_a \le \min_{v_h \in S^h} \|z(s) - v(s)\|_a, \forall\, v(s) \in S^h, \qquad [27]$$

Proof. Recall orthogonality property, we have

$$\|z - Z_h\|_a^2 = (z - Z_h, z - Z_\mathrm{h})_a$$

$$= (z - Z_h, z\,)_\alpha - (z - Z_h, Z_h)_a$$

$$= (z - Z_h, z\,)_a$$

$$= (z - Z_h, z\,)_\alpha - (z - Z_h, Z_h\,)_a$$

$$= (z - Z_h, z - v_h\,)_a \quad \forall v_h \in V_h.$$

hence, by the Cauchy-schwarz inequality,

$$\|z - Z_h\|_a^2 \le \|z(s) - Z_h(s)\|_a \|z(s) - \mathrm{v}(s)\|_a \quad \forall v_h \in V_h,$$

therefore

$$\|z - Z_h\| \le \|z - v_h\| \quad \forall v_h \in V_h.$$

Consequently,

$$\|z(s) - Z_h(s)\|_a = \min_{v \in V_h} \|z(s) - v_h(s)\|_a.$$

Theorem 4.1. [An a priori error estimate] Let $z(s)$and $Z_h(s)$ be the solutions of the 1 and the finite element problem (FEM), respectively. Then there exists an interpolation constant Ci, depending only on a(x), such that

$$\|z(s) - Z_h(s)\|_a \leq C\left\|h^2 z^{(4)}\right\|_a$$

Proof. Since $\Pi_h z(x) \in S^h$, we may take $v = \Pi_h z(x)$ in (27), gives

$$\|z(s) - Z_h(s)\|_a \leq C\|z(s) - \Pi_h z(s)\|_a$$

$$= \left(\int_0^1 C(s)(z'(s) - (\pi_h z)'(s))^{\,2} ds + (\mathrm{z}(s) - \pi_h z(s))^{\,2}\right)^{\frac{1}{2}}$$

$$\leq C_1(\|z'(s) - (\pi_h z)'(s)\|_{L^2} + \|z(s) - \Pi_h z(s)\|_{L^2})$$

where $C_1 = max\{a^{1/2}, 1\}$. The proof will be finished through (Rüdiger, 2013).

5. Numerical Experiments

We present in this section a numerical experiment aiming to investigate the performance of the presented modified cubic B spline finite element method for boundary value problems through Mathematica programming. Different examples are given and the accuracy of the present method is measured by the errors

Absolute error=$\|Z_h - z\|$

$$L_2 = \|Z_h - z\|_{L^2} = \sqrt{h\sum_{j=0}^{n}\left\|Z_{j,h} - z_{j,h}\right\|^2}\ , \quad L^\infty = \|Z_h - z\|_{L^\infty} = \max_j \left\|Z_{j,h} - \mathrm{z}_{\mathrm{j,h}}\right\|,$$

and the order of convergence is calculated by Rate $=\frac{\ln\frac{\text{error 1}}{\text{error 2}}}{\ln\frac{\text{h1}}{\text{h2}}}$

Example (5.1): We consider the following boundary value problem.

$$-\mathrm{z}'' + 2\mathrm{z} = \mathrm{s}^2\mathrm{e}^\mathrm{s} - 5\mathrm{s}\mathrm{e}^\mathrm{s},$$

with boundary conditions

$$\mathrm{z}(0) = 0\,;\mathrm{z}(1) = 0.$$

The exact solution is $\mathrm{z}(\mathrm{s}) = \mathrm{s}(\mathrm{s}-1)\mathrm{e}^\mathrm{s}.$

Table 2: Comparison between exact and numerical solutions for example (5.1)

s	Cubic B-spline	Exact solution	Absolute error
0.1	-0.09946419	-0.09946538	1.19 e-06
0.2	-0.19542276	-0.19542444	1.68 e-06
0.3	-0.28346849	-0.28347035	1.86 e-06
0.4	-0.35803564	-0.35803793	2.29 e-06
0.5	-0.41217768	-0.41218032	2.64 e-06
0.6	-0.43730528	-0.43730851	3.23 e-06
0.7	-0.42288450	-0.42288807	3.57 e-06
0.8	-0.35608182	-0.35608655	4.73 e-06
0.9	-0.22135963	-0.22136428	4.65 e-06

Table 3: Rate of convergence for example 5.1

h	L_∞ error	Rate	L_2 error	Rate
1/3	4.04108 e-04		2.71753 e-04	
1/6	3.21604 e-05	3.65138	2.09924 e-05	3.69436
1/12	2.40315e-06	3.74228	1.45043 e-06	3.85532
1/24	1.71256e-06	3.8107	9.50565 e-06	3.93155
1/48	9.42981e-06	4.18278	4.53488 e-06	4.38965

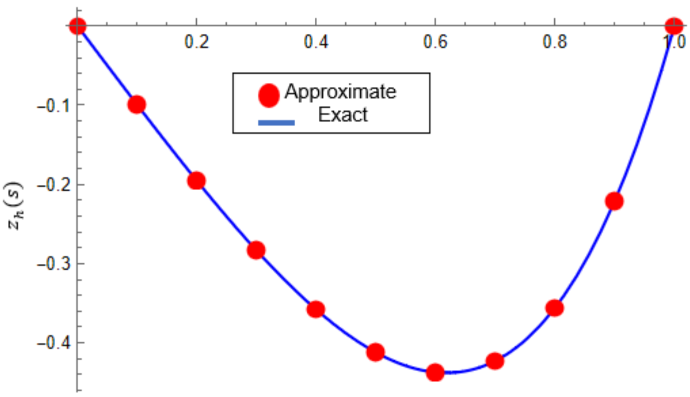

Figure 1: Example 5.1 Comparison between exact and numerical solutions

Example (5.2): We consider the following boundary value problem.

$$-z'' + 2z = -6s + 2s^3,$$

with boundary conditions

$$z(0) = 0\,; z(1) = 1.$$

The analytical solution is $z(s) = s^3$

We can get the coefficient are given bellow, for $h = 0.1$

Table 3: Comparison between exact and numerical solutions for example (5.2)

s	Cubic B-spline	Exact solution	Absolute error
0.1	0.001	0.001	1.67906 e-12
0.2	0.008	0.008	3.36813 e-12
0.3	0.027	0.027	5.13627 e-12
0.4	0.064	0.064	6.89422 e-12
0.5	0.125	0.125	9.22351 e-12
0.6	0.216	0.216	1.12548 e-11
0.7	0.343	0.343	1.56503 e-11
0.8	0.512	0.512	1.25325 e-11
0.9	0.729	0.729	7.48879 e-11

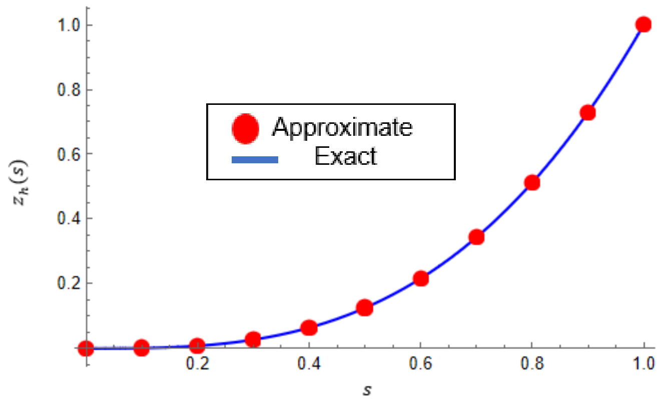

Figure 2: Example 5.2 Comparison between exact and numerical solutions

Example 5.3: consider the boundary value problem.

$$-z'' + 2z = \sin(\pi s),$$

with boundary conditions

$$z(0) = 0\,; z(1) = 0.$$

The analytical solution is $z(s) = \frac{\sin(\pi s)}{(2+\pi^2)}$

Table 4: Comparison between exact and numerical solutions for example 5.3

s	Cubic B-spline	Exact solution	Absolute error
0.1	0.0260346	0.0260343	3.11507 e-07
0.2	0.0495209	0.0495202	7.07491 e-07
0.3	0.0681596	0.0681587	9.31793 e-07
0.4	0.0801265	0.0801254	1.11257 e-07
0.5	0.0842500	0.0842488	1.16078 e-07
0.6	0.0801265	0.0801254	1.11257 e-07
0.7	0.0681596	0.0681587	9.31795 e-07
0.8	0.0495209	0.0495202	7.07492 e-07
0.9	0.0260346	0.0260343	3.11508 e-07

Table 6: Rate convergence for example (5.3)

h	L_∞ error	Rate	L_2 error	Rate
1/3	1.40747 e-04		1.1492 e-04	
1/6	9.05776 e-06	3.95781	6.5384 e-06	4.13554
1/12	5.5856 e-07	4.01937	3.94087 e-07	4.05235
1/24	3.44535 e-08	4.01899	2.43635 e-08	4.01572
1/48	2.51274 e-09	3.77732	1.67108 e-09	3.86587

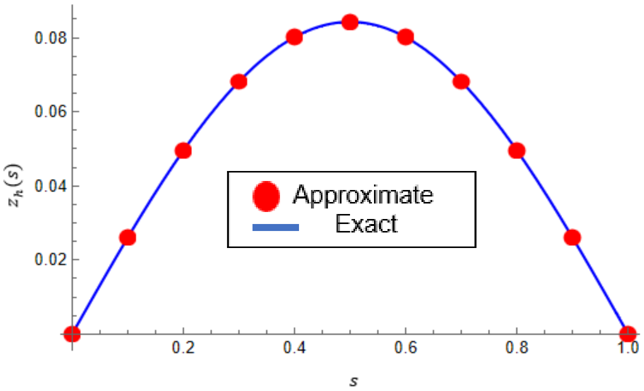

Figure 3: Example 5.3 Comparison between exact and numerical solutions

Example (5.4): We solve the following boundary value problem.

$$z'' - z' = -1 - e^{s-1},$$

with boundary conditions

$$z(0) = 0\,; z(1) = 0.$$

The analytical solution is $z(s) = s - se^{s-1}$

We can get the coefficient are given bellow, for $h = 0.1$

Table 7: Comparison between exact and numerical solutions for example 5.4

s	Cubic B-spline	Exact solution	Absolute error
0.1	0.0593428	0. 0.059343	2.07776 e-07
0.2	0.110134	0.110134	2.76097 e-07
0.3	0.151024	0.151024	2.88773 e-07
0.4	0.180475	0.180475	3.38836 e-07
0.5	0.196734	0.196735	3.73154 e-07
0.6	0.197808	0.197808	4.36314 e-07
0.7	0.181427	0.181427	4.62378 e-07
0.8	0.145015	0.145015	5.88055 e-07
0.9	0.0856458	0.0856463	5.51441 e-07

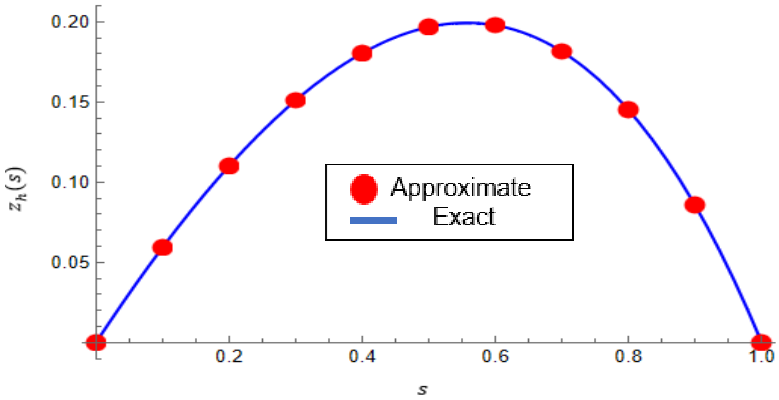

Figure 4: Example 5.4 Comparison between exact and numerical solutions

Table 8: Numerical results of Example (5.4)

s	CBS colocation (Munguia & Bhatta, 2015)	FDM (Ahmed, 2017)	Presented method
0.1	8.27 e-05	4.10362 e-04	2.07776 e-07
0.2	1.34 e-04	7.8888 e-05	2.76097 e-07
0.3	2. e-04	1.12030 e-04	2.88773 e-07
0.4	1.17 e-04	1.38578 e-04	3.38836 e-07
0.5	2.833 e-04	1.56368 e-04	3.73154 e-07
0.6	2.3334 e-04	1.62719 e-04	4.36314 e-07
0.7	2.5523 e-04	1.54431 e-04	4.62378 e-07
0.8	2. e-04	1.27683 e-04	5.88055 e-07
0.9	1.1 e-04	7.7993 e-05	5.51441 e-07

Table 9: Rate convergence for example (5.4)

h	L_∞ error	Rate	L_2 error	Rate
1/3	0.0954706	…….	0.0850279	……..
1/6	1.75372 e-05	12.4104	1.29017 e-05	12.6862
1/12	1.35467 e-06	3.69441	9.25992 e-07	3.80042
1/24	9.7922 e-08	3.79016	6.14624 e-08	3.91322
1/48	6.51026 e-09	3.91085	4.16694 e-09	3.88264

In all tables, it is seen that there is a good agreement between approximate and exact solutions that is mesh size decreases In Table 8, we have compared errors with CBS colocation and finite difference methods in (Jameel, 2020). As it is seen from the table, the results obtained in the present study is better than in other studies such as colocation and finite difference, the error decreases. In tables 3, 6 and 9 shows that the rate of convergence is order four which confirms with theoretical results.

6. Conclusion

This work aims to apply modified cubic B spline finite element method in approximating numerical solutions for ordinary differential equations. Error analysis for the proposed scheme are analysed through Cea's lemma. The suggested method is proved in term of fourth order in space. Furthermore, the behavior of the exact solution and approximate solution are examined graphically. The numerical results obtained by the presented method are quite satisfactory from the exact solution.

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