Application of Lagrange Interpolation Method to Solve First-Order Differential Equation Using Newton Interpolation Approach

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Abstract: One of the important problems in mathematics is finding the analytic solution and numerical solution of the differential equation using various methods and techniques. Most of the researchers tackled different numerical approaches to solve ordinary differential equations. These methods such as the Runge Kutta method, Euler's method, and Taylor's polynomial method have so many issues like difficulties in finding the solution that can lead to singularities or no solution. In this work, we considered Newton's interpolation and Lagrange's interpolation polynomial method (LIPM). These studies combine both Newton's interpolation method and Lagrange method (NIPM) to solve first-order differential equations. The results obtained provide minimum approximative error. The result is supported by solving an example.

Keywords: Differential Equation, Lagrange Interpolation Method, Newton Interpolation First-order Differential Equation

1. Introduction

A differential equation is a mathematical statement that compares the derivatives of one or more variables. Functions that indicate physical values, and derivatives explain their rates of change, and differential equations define the relationship between the differences in most situations. Differential equation plays a critical part in many areas, including physics, engineering, biology, and economics, because such interactions are frequent (Zill, 2012). The language in which the laws of nature are described is differential equations. Understanding the concept of differential equation solutions is essential to most modern science and engineering. The ordinary differential equations (ODEs) deal with derivatives of a single variable, which is sometimes called time. The solution to first-order ODEs using analytical, graphical, and numerical approaches is one of the issues discussed. ODEs have constant variables, typically second-order ODEs; Variation of variables and unknown variables; Variations, damping, reflection in sinusoidal, exponential signals; Complex numbers and exponentials; Delta functions, convolution, and Laplace transform techniques; Fourier line, periodic solutions; Fourier series, periodic equation and imaginary part in matrices and first-order linear systems; and critical point analysis and phase plane drawings for nonlinear intelligent devices

Many issues in real life may be represented in the form of ODEs, particularly those of first degree. To solve numerical issues, a numerical method is used.

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Several methods for finding numerical solutions to ODEs are studied (Faith, & Seeam, 2018; Cuneyt, Cansun, Ibrahim, Tuncay, Oktay, 2004; Ide, 2008; Islam, Khan, Faraz, & Austin, 2010; Kreuzberger, et al., 2020; Weber, 1975) differential equations (ODEs), typically the first and 2nd - order equations, and indeed the ability to solve differential calculus is critical. The study of exact and projected numerical methods to nonlinear finite difference methods of first and second-order is becoming increasingly important in a variety of physical-mathematical sciences, including applied mathematics, mechatronics, electrical engineering, computational physics, summarized matter physics, particle physics, and, as a result, particle physics. (Islam, & Akbar, 2020). Differential equations are among the most important subjects in mathematics, with a number of different approaches and solutions. We have research techniques and numerical methods; different techniques are only effective for a limited set of equations; hence numerical approaches are often employed. (Faith, 2018).

But the problem of finding a solution to some difficult differential equations has been a major problem for scientists to deal with, tackling these kinds of problems necessitates the authors in [Ibrahim, 2020; Ibrahim, & Isah, 2021; Ibrahim, & Isah, 2022; Isah, & Ibrahim, 2021; Salisu, 2022b] to introduced a numerical method for solving ODEs, partial differential equations (PDEs), and fractional differential equations (FDEs). Therefore, commutativity is very important from a practical point of view.

The authors in (Ibrahim, & Koksal, 2021a) studied the commutativity with non-zero initial conditions (ICs) and their effects on the sensitivity was studied in [Salisu, I, 2022a; Salisu, I, 2022c] while the realization and decomposition of a fourth-order LTVSs with nonzero ICs by cascaded two Second-Order commutative pairs was introduced by (Ibrahim, & Koksal, 2021b; Salisu, & Rababah, 2022). The authors in (Rababah, & Ibrahim, 2016a; Rababah, & Ibrahim, 2016b; Rababah, & Ibrahim, 2018) come up with a numerical approximative process for degree reduction of curves and surface which approaches can be used to solve complex ODES, PDEs, and FDEs.

Newton's interpolation is used extensively in numerical analysis and picture processing. This study introduces univariate and bivariate parameterized Newton-type polynomial interpolation algorithms. Because the proposed new interpolation functions are parametric, they are not information.”(Zou, et al., 2020; Zhao, et al 2021).

Lagrange's equations include a systematic approach to defining the differential equation of a mechanical device or a (flexible) main structural with many degrees of freedom. By stating the scalar quantities, a scalar method is produced in basis of geometric parameters of energy of a system *(Sauer, 2004)*. This report's discussion of Lagrange's approach is limited, yet it contains enough information. Background information for the flexible structure's vibration signals in the laboratory course for the mean absolute error. The above study will look closely at differentiation and solve a first and second order differential equations using Newton interpolation and the Lagrange interpolation approach. The purpose of this work is to find the numerical approximation of the first and second order differential equations using Newton interpolation and the Lagrange interpolation approach.

2. Preliminaries

In this section we introduce the first-order differential equation, second-order differential equation and Euler’s method for solving the first and second-order differential equation, we also show the formula of Taylor polynomial and Runge-Kutta method for solving ODEs, PDEs and FDEs, all this are numerical techniques that proof to be the best numerical method for finding the solution of first and second differential equation, but in this work we are going to consider Lagrange interpolation polynomial and Newton interpolation method to solve first-order ODEs, which will be explain the next sectional.

Considering the first and second-order systems described as

$$\boldsymbol{y}'(x) + Q(x)y(x) = \mathrm{f}(x), \quad [1]$$

$$y''_B(x) + \mathrm{P}(x)\boldsymbol{y}'(x) + \mathrm{Q}(x)y(x) = \mathrm{f}(x), \quad [2]$$

where $\mathrm{P}(x)$, $\mathrm{Q}(x)$ and $\mathrm{f}(x)$ are function of x.

2.1 Euler Method

Euler's method is a developed numerical solution to an initial value problem of the type

$$y'(t)f(x, y), \quad [3]$$

$$y(x_o) = y_0. \quad [4]$$

Beginning with the initial condition y_0, we create the rest of the solution using repeated formulas. To develop a numerical solution to an initial value issue of the form:

$$x_{n+1} = x_n + h, \quad [5]$$

$$y_{n+1} = y_n + hf(x_n, y_n), \quad [6]$$

where h is the step function, x_{n+1} is the independent value that can be divided into h sub intervals and y_{n+1} is the solution numerical solution.

2.2 Taylor Series

The Taylor series can be defined as

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 +, \ldots \quad [7]$$

To determine a condition that must be true for a Taylor series to exist for a function, we first construct the nth degree Taylor mathematical model of that function.

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i. \quad [8]$$

This polynomial has a maximum degree of n and it is called the Taylor polynomial.

2.3 Runge-Kutta Method

We do have deferential expression as first, second, third, fourth and so on of Runge-kutta methods as yield in Eq. (9)

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \quad [9]$$

$$k_4 = hf(x_n + h, y_n + k_3),$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

3. Material and Methods

3.1 Lagrange Interpolation Polynomial

The LIPM is numerical approximation that involve interval of a given endpoints.

Theorem 1: If $x_0, x_1, ..., x_n$ are $n+1$ difference numbers and $f(x)$ is a function, then there exist a unique polynomial $P(x)$ of degree at most n with

$$f(x_k) = P(x_k), \quad for\ each\ \ k = 0, 1, \dots, n. \qquad [10]$$

And

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \qquad [11]$$

where, for each k = 0, 1, ..., n,

$$L_{n,k}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}. \qquad [12]$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

By substituting $n = 3$ in Eq. (11), we obtained the Lagrange interpolating polynomials by the use of Mathematica software through four points:

$$P(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)}y_0 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)}y_1 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)}y_2 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)}y_3.$$

3.2 Newton's Interpolation Method

Newton interpolation is a quadratic interpolation methodology used in numerical methods and outcomes. The interpolation formula in most classic procedures is particular to the data. This paper discusses single and multivariable generalized Newton type polynomial interpolation approaches.

The forward difference formula and the backward difference formula are used in Newton polynomial interpolation.

$$y_0(x) = a_0, \qquad [13]$$

$$y_1(x) = a_0 + a_1(x - x_0), \qquad [14]$$

$$y_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1), \qquad [15]$$

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0) + \cdots \dots . + a_n(x - x_0)(x - x_1) \dots \dots \dots (x - x_{n-1}), \qquad [16]$$

where

$$a_0 = y_0, \quad [17]$$

$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}, \quad [18]$$

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{(x_2 - x_0)}, \quad [19]$$

$$a_3 = \frac{\frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1} - \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}}{x_3 - x_0}, \quad [20]$$

$$a_n = f[x_k, x_{k-1}, \dots, x_1, x_0] = \frac{f[x_k, x_{k-1}, \dots, x_2, x_1] - f[x_{k-1}, x_{k-2}, \dots, x_1, x_0]}{x_k - x_0}. \quad [21]$$

3.3 Forward Difference Table

The value inside the boxes of the following difference in Table 1 is used in deriving the newton forward difference interpolation formula by setting $x = x_0 + ph$ and $a_0, a_{1,\cdots a_n}$, the newton forward difference equation is given as

$$p_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2i}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3i}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{ni}\Delta^n y_0. \quad [22]$$

Table 1: Forward difference interpolation formula

Value of x	Value of $y = f(x)$	First Difference $\Delta f(x)$	Second Difference $\Delta^2 f(x)$	Third Difference $\Delta^3 f(x)$	Fourth Difference $\Delta^4 f(x)$
x_0	y_0				
		Δy_0			
$x_0 + h$	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
$x_0 + 3h$	y_3		$\Delta^2 y_2$		
		Δy_3			
$x_0 + 4h$	y_4				

3.4 Backward Difference Table

The value inside the boxes of the following difference Table 2 is used in deriving the newton backward difference interpolation formula which is given as

$$p_n(x) = y_n + p\nabla y_0 + \frac{p(p+1)}{2i}\nabla^2 y_n + \frac{p(p+1)(p-2)}{3i}\nabla^3 y_0 + \cdots$$
$$+ \frac{p(p+1)(p-2)\cdots(p+n-1)}{ni}\nabla^n y_n. \quad [23]$$

Table 2: Backward difference interpolation formula

Value of x	Value of $y=f(x)$	First Difference $\nabla f(x)$	Second Difference $\nabla^2 f(x)$	Third Difference $\nabla^3 f(x)$	Fourth Difference $\nabla^4 f(x)$
x_0	y_0				
		∇y_1			
x_0+h	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_0+2h	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\Delta^3 y_4$	
x_0+3h	y_3		∇y_4		
		∇y_4			
x_0+4h	y_4				

4. Application

In this section, we make use of the formula and conditions obtained from the previous section and illustrate the numerical solution of first-order differential equations.

Example 1. Let us first consider the following first-order differential equations

$$cosx - x^2 y = f(x) \qquad y(0) = 0 \qquad h = 0.01. \quad [24]$$

By applying the Newton interpolation of Eqs. (13-21), we obtain the following

$$a_o = 0 = y_o, \quad [25]$$

$$a_1 = \frac{f(x_1) - f(x_o)}{x_1 - x_o} = \left[\frac{dy}{dx}\right]_{0,0} = cos(0) - (0)^2(0) = 1 - 0 = 1 \quad [26]$$

$$y_1 = a_o + a_1(x - x_o) = 0 + 1(0.01 - 0) = 0.01 \quad [27]$$

$$a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_o)}{x_1 - x_o}}{x_2 - x_o} = \frac{\left[\frac{dy}{dx}\right]_{0.01,\ 0.01} - \left[\frac{dy}{dx}\right]_{0,0}}{0.02 - 0}$$

$$a_2 = \frac{cos(0.01) - (0.01)^2(0.01) - (cos(0) - (0)^2(1))}{0.02 - 0} = \frac{0.9999492 - 1}{0.02} = -0.00254998 \quad [28]$$

$$y_2 = 0 + 1(0.02 - 0) + (-0.00254998)(0.02 - 0)(0.02 - 0.01)$$
$$= 0.02 - 5.09996 \times 10^{-7} = 0.0199995 \quad [29]$$

By substituting $n = 2$ in the Lagrange interpolation polynomial in Eq. (11), we obtain

$$y_2 = \frac{(x-x_1)(x-x_2)}{(x_o-x_1)(x_o-x_2)}(y_o) + \frac{(x-x_o)(x-x_2)}{(x_1-x_o)(x_1-x_2)}(y_1) + \frac{(x-x_o)(x-x_1)}{(x_2-x_o)(x_2-x_1)}(y_2), \quad [30]$$

Where

$$x_o = 0, \;\; x_1 = 0.01, \;\; x_2 = 0.02, \quad [31]$$

Considering the values x_o, x_1, x_2 of Eq. (31) and y_o, y_1, y_2 of Eq. (25), Eq. (27), and Eq. (29) respectively, and inserting them in Eq. (30), we obtain

$$y_2 = \frac{(x-0.01)(x-0.02)}{(0-0.01)(0-0.02)}(0) + \frac{(x-o)(x-0.02)}{(0.01-0)(0.01-0.02)}(0.01) + \frac{(x-o)(x-0.01)}{(0.02-0)(0.02-0.01)}(0.1999)$$

$$= \frac{x^2-0.02x}{1\times10^{-4}}(0.01) + \frac{x^2-0.01x}{2\times10^{-4}}(0.19999999) = -0.00254998x^2 + 1.00003x. \quad [32]$$

The exact solution is given by

$$y_{exact} = \frac{1}{2}e^{\frac{1}{2}-\frac{x^2}{2}}\sqrt{\frac{\pi}{2}}\left(\text{Erfi}\left[\frac{-i+x}{\sqrt{2}}\right] + \text{Erfi}\left[\frac{i+x}{\sqrt{2}}\right]\right). \quad [33]$$

The figures below depict the graph of approximate solution with exact solutions and error between.

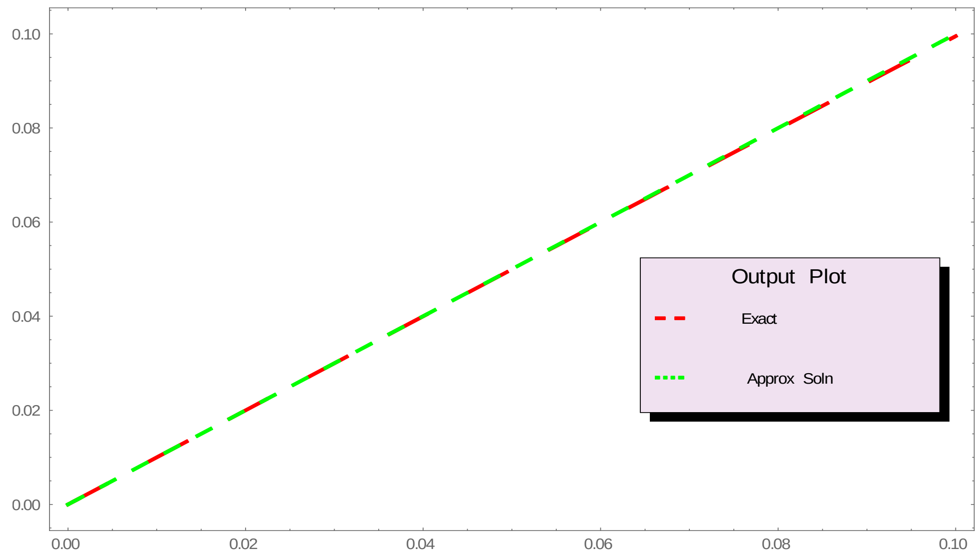

Figure 4.1: Solution of Example 1 with LIPM and NIPM.

The error is defined as

$$error = y_{exact} - y_2 \quad [4.13]$$

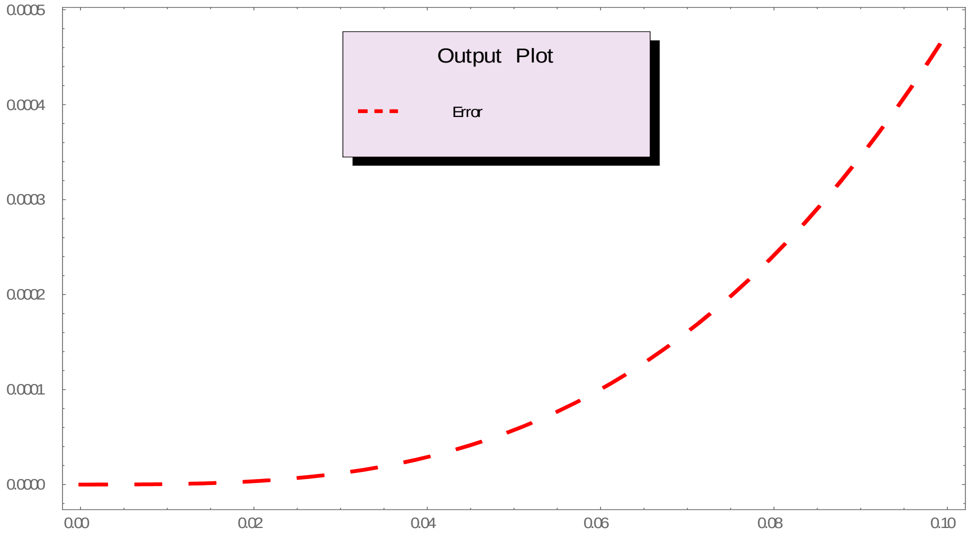

Figure 4.2: Error of Example 1 with LIPM and NIPM

Table 4.2: The Table showing the result of Example 1

x	Exact Values	Newtons & Langrage.	Errors
0	0	0	0
0.01	0.0099995	0.01	4.99989×10^{-7}
0.02	0.019996	0.0199995	3.48966×10^{-6}
0.03	0.0299865	0.0299985	0.0000119674
0.04	0.039968	0.0399969	0.0000289289
0.05	0.0499375	0.0499949	0.0000573662
0.06	0.0598921	0.0599924	0.000100266
0.07	0.0698287	0.0699893	0.000160608
0.08	0.0797444	0.0799857	0.0002413655
0.09	0.0896361	0.0899816	0.000345501
0.1	0.0995011	0.0999771	0.000475968

5. Conclusion

This paper studies the solution of first-order differential equations using the langrage interpolation polynomial method and Newton interpolation approached. The result obtained shows that the proposed method and approached gives a minimal approximation error and outperforms the existing methods. The numerical results are verified to be correct by an example that is computed using Mathematica and MATLAB.

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