Hybrid Method for Accretive Variational Inequalities Involving Pseudocontraction

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Abstract: We use strongly pseudocontractions to regularise a class of accretive variational inequalities in more general settings, the solutions are sought in the set of fixed points of another pseudocontraction. In this paper, we consider an implicit scheme that can be used to find a solution of a class of accretive variational inequalities. Our results improved and generalise some results of Yaqin and Chen.

Keywords: Pseudocontractions, Variational Inequalities, Monotone, Accretive

1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = \| x \| \| f \|, \| x \| = \| f \|\}, \forall\, x \in E$
where E^* denotes the dual space of E and $\langle .,.\rangle$ denotes the generalized duality pairing. In what follows, we shall denote the single-valued duality mapping by j and $Fix(T) = \{x \in E : Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x$) will denote strong (respectively weak) convergence of x_n to x.
A mapping T with domain D(T) and range R(T) in E is said to be pseudocontractive if the inequality

$$\| x - y \| \leq \| x - y + t((I-T)x - (I-T)y \| \tag{1}$$

holds for each $x, y \in D(T)$ and for all $t > 0$. It is very easy to understand that (1) is equivalent to (2) below if there exist $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \| x - y \|^2 \tag{2}$$

for any $x, y \in D(T)$. T is called strongly pseudocontractive if there exist
$j(x-y) \in J(x-y)$ and $\eta \in (0,1)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \eta \| x - y \|^2$ for any $x, y \in D(T)$.
Let H be a Hilbert space with inner product $\langle .,.\rangle$, we know that $T : C \to H$ is called monotone if $\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in C$. A variational inequality problem,denoted by VI(T, C) is to find a point x^* with property

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$$x^* \in C \quad such \quad that \quad \langle Tx^*, x - x^* \rangle \geq 0 \quad \forall x \in C$$

If the mapping T is monotone operator,then we say that VI(T,C) is monotone.
Lu et al.(Ceng, Lin, & Petruşel, 2012) considered the following monotone variational inequality problem in Hilbert spaces (denoted by VI(TS,C))

$$find \quad x^* \in Fix(T) \quad such \quad that \quad \langle (I-S)x^*, x - x^* \rangle \geq 0 \quad \forall x \in Fix(T). \tag{3}$$

Where $T,S : C \to C$ are non expansive mappings and $Fix(T) \neq \emptyset$.
Yao et al.(Yao, Marino, & Liou, 2011) considered VI(TS,C) in Hilbert spaces when $T,S : C \to C$ are pseudocontraction. Recently, Wang and Chen consider the following variational inequality problem in Banach space

$$find \quad x^* \in F(T) \quad such \quad that \quad \langle (I-S)x^*, j(x - x^*) \rangle \geq 0 \quad \forall x \in F(T). \tag{4}$$

Where $T,S : C \to C$ are pseudocontraction. Since $(I-S)$ is accretive the Variational in equality (4) is accretive.
For solving the VI(T, C), hybrid methods were studied by Yamada (Yamada & Ogura, 2005) where he assumed that T is Lipschitzian and strongly monotone. However, his methods do not apply to the Variational inequality (4) since the mapping $(I-S)$ fails, in general, to be strongly monotone, though it is Lipschitzian. In fact the Variational inequality (4) is, in general, ill-posed, and thus regularization is needed.

Let $T,S : C \to C$ be non expansive and $f : C \to C$ be contractive. In 2006 Moudafi and Mainge (Moudafi & Maingé, 2006) studied the VI(TS,C) by regularizing the mapping $tS + (1-t)T$ and defined $\{x_{s,t}\}$ as follows,

$$x_{s,t} = sf(x_{s,t}) + (1-s)[tSx_{s,t} + (1-t)Tx_{s,t}], \quad s,t \in (0,1)$$

Since Moudafi and Mainge's regularization depends on t, the convergence of the scheme above is more complicated, so Lu et al.Ceng et al. (2012) defined $\{x_{s,t}\}$ as the unique fixed point of the equation

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)Sx_{s,t}] + (1-s)Tx_{s,t} \quad s,t \in (0,1) \tag{5}$$

Note that Lu et al's regularization does no longer depend on t, and their result for the regularizing (5) is under less restrictive conditions than Moudafi and Mainge's.
Yao in Yao et al. (2011) extended Lu et al.'s result to a general case, i.e., in the scheme (5), S, T are extended to Lipschitz pseudocontractive and f is extended to strongly pseudocontractive. in 2011 Wang and Chen (2011) observed that a continuity condition on f is necessary, so, they modify it. Further they used strongly pseudocontraction to regularize the ill-posed accretive variational inequality (4). and proved the convergence of the scheme (5) in Banach spaces that admit weakly sequencially duality mapping.
Motivated by the above work in this paper we prove and analyse the convergence of the scheme (5) in more general setting that involve the spaces that do not admit weakly sequencially continous duality mapping. Our result improve and extend the corresponding results.(Ceng et al., 2012; Wang & Chen, 2011; Yao et al., 2011)

2. Preliminaries

A Banach space E is said to be uniformly convex if given $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that $\forall\, x,y \in E$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$, we have $\|\frac{x+y}{2}\| \leq 1-\delta$.
E is strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x,y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $S(E) := \{x \in E : ||x|| = 1\}$ be a unit sphere of E, then E is said to have Gˆateaux differentiable norm (or E is said to be smooth), if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|y\|}{t}$$

exists for all $x,y \in S(E)$. E also have uniformly Gˆateaux differentiable norm (or its uniformly smooth) if the limit exists uniformly for $x,y \in S(E)$.
It is well known that if E smooth, then the normalised duality map J is single-valued. Also if E is uniformly smooth, then the normalised duality map J is $norm-to-weak^*$ uniformly continuos on bounded subsets of E.

Let μ be a continous linear functional on l^∞ and let $(a_1, a_2, ...) \in l^\infty$ we write $\mu_k(a_k)$ instead of $\mu(a_1, a_2, ...)$. We recall that μ is a Banach limit when it satisfies $||\mu|| = \mu_k(1) = 1$ and $\mu_k(a_{k+1}) = \mu_k(a_k)$ for each $(a_1, a_2, ...) \in l^\infty$. if μ is a Banach limit, then we have the following:
(i) $\forall n \geq 1, \quad a_n \leq c_n$ implies $\mu(a_n) \leq \mu(c_n)$
(ii) $\mu(a_{n+r}) = \mu(a_n)$
(iii)$\liminf_{n\to\infty}(a_n) \leq \mu(a_n) \leq \limsup_{n\to\infty}(a_n)$.
Recall that $S : C \to C$ is called accretive if $I-S$ is pseudocontractive. We denote by J_r the resolvent of S i.e $J_r = (I+rS)^{-1}$.

It is well known that J_r is nonexpansive, single valued and $Fix(J_r) = S^{-1}(0) = \{z \in D(S) : 0 = Sz\} \quad \forall r > 0$.
Now let $T : C \to C$ be a pseudocontractive mapping, then, $I - T$ is accretive we denote $A = J_1 = (2I - T)^{-1}$. Then Fix(A) = Fix(T) and $A : R(2I - T) \to C$ is non expansive. The following lemma can be found in Song and Chen (2007); Wang and Chen (2011)

Lemma 0.1. *Let C be a non empty closed convex subset of a real Banach space E and $T : C \to C$ be a continous pseudocontractive map, we denote by $A = J_1 = (2I - T)^{-1}$ then,*
(i) The map A is non expansive self mapping on C.
(ii) If $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ then $\lim_{n\to\infty} ||x_n - Ax_n|| = 0$.

Lemma 0.2. *Let C be a non empty closed convex subset of a uniformly convex space E and let $T : C \to C$ be non expansive then $I - T$ is demiclosed on C*

We also need the following lemma

Lemma 0.3. *Let C be a non empty closed convex subset of a real Banach space E. Assume that $F : C \to E$ is accretive and weakly continous along the segments; that is $F(x + ty) \rightharpoonup F(x) \quad as \quad t \to 0$. Then, the variational inequality*

$$x^* \in C, \langle Fx^*, j(x - x^*)\rangle \geq 0 \quad \forall x \in C \tag{6}$$

is equivalent to the dual variational inequality

$$x^* \in C, \langle Fx, j(x - x^*)\rangle \geq 0 \quad \forall x \in C \tag{7}$$

The following lemma is usefull for the proof of our main result

Lemma 0.4. *Let E be real normed linear space then the following inequality holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle \quad \forall x, y \in E \quad j(x + y) \in J(x + y)$$

3. Main Results

Let C be a nonempty closed convex subset of a real Banach space E. Let $f : C \to C$ be a Lipschitz strongly pseudocontraction and $T, S : C \to C$ be two continuous pseudocontractions. For $s, t \in (0, 1)$, we define the following mapping

$$x \mapsto W_{s,t} := s[tf(x) + (1 - t)Sx] + (1 - s)Tx$$

It is easy to see that the mapping $W_{s,t} : C \to C$ is a continuous strongly pseudocontractive mapping. So, by (Jung, 2005), $W_{s,t}$ has a unique fixed point which is denoted $x_{s,t} \in C$; that is

$$x_{s,t} = W_{s,t} = s[tf(x) + (1 - t)Sx] + (1 - s)Tx, \quad s, t \in (0, 1) \tag{8}$$

Theorem 0.5. *Let E be a real reflexive and strictly convex Banach space with uniformly Gateaux differentiable norm. Let C be a nonempty closed convex subset of E. Let $f : C \to C$ be a Lipschitz strongly pseudocontraction, $S : C \to C$ be a Lipschitz pseudocontraction and $T : C \to C$ be a continous pseudocontraction with $Fix(T) \neq \emptyset$. Suppose that the solution set Ω of the Variational inequality (4) is nonempty. Let for each $(s, t) \in (0, 1)^2$, $\{x_{s,t}\}$ be defined by (8). Then for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a point $x_t \in Fix(T)$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution x^* of the following variational inequality*

$$x^* \in \Omega, \langle (I - f)x^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in \Omega. \tag{9}$$

Hence, for each null sequence $\{t_n\} \in (0, 1)$, there exist another null sequence $\{s_n\} \in (0, 1)$, such that the squence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Proof. First we show that for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ is bounded.
For any $z \in fix(T), \quad \forall s, t, \in (0, 1)$, by (8) we have
$||x_{s,t} - z|| = \langle s[tf(x) + (1 - t)Sx] + (1 - s)Tx - z, j(x_{s,t} - z)\rangle$

$$\begin{aligned}
&= st\langle f(x_{s,t} - f(z), j(x_{s,t} - z)\rangle + s(1 - t)\langle Sx_{s,t} - Sz, j(x_{s,t} - z)\rangle \\
&+(1 - s)\langle Tx_{s,t} - Tz, j(x_{s,t} - z)\rangle + st\langle f(z) - z, j(x_{s,t} - z) \\
&+s(1 - t)\langle Sz - z, j(x_{s,t} - z)\rangle \\
&\leq st\beta||x_{s,t} - z||^2 + s(1 - t)||x_{s,t} - z||^2 + (1 - s)||x_{s,t} - z||^2 \\
&st||f(z) - z||||x_{s,t} - z|| + s(1 - t)||Sz - z||||x_{s,t} - z||
\end{aligned}$$

$$= (1 - st(1-\beta))||x_{s,t} - z||^2 + s[t||f(z) - z|| + (1-t)||Sz - z||]||x_{s,t} - z||,$$

which implies that

$$||x_{s,t} - z|| \leq \frac{t||f(z) - z||}{t(1-\beta)} + \frac{(1-t)||Sz - z||}{t(1-\beta)}$$
$$\leq \frac{1}{t(1-\beta)} max\{||f(z) - z||, ||Sz - z||\}$$

Hence for each $t \in (0,1)$, $\{x_{s,t}\}$ is bounded. Further, by the Lipchitz continuity of f and S, we have that $\{f(x_{s,t})\}$ and $\{S(x_{s,t})\}$ are both bounded for each $t \in (0,1)$. From (8), we also have

$$||Tx_{s,t}|| \leq \frac{1}{1-s}||x_{s,t} + \frac{s}{1-s}||tf(x_{s,t}) + (1-t)Sx_{s,t}||$$

So $\{Tx_{s,t}\}$ is also bounded as $s \to 0$ for each $t \in (0,1)$
Now, we show $x_{s,t} \to x_t \in Fix(t)$ as $s \to 0$
From (8) we have that
$x_{s,t} - Tx_{s,t} = s[tf(x_{s,t}) + (1-t)Sx_{s,t} - Tx_{s,t}] \to 0$ as $s \to 0$
Let $\{s_n\}$ be a null sequence in (0,1) such that $x_{s_n,t}$ satisfies (8)
Define a map $\varphi : E \to \Re$ by:

$$\varphi(y) = \mu||x_{s_n,t} - y||^2 \quad \forall y \in E$$

Then, clearly $\varphi(y) \to \infty$ as $||y|| \to \infty$, φ is continous and convex, so as E is reflexive there exist $q \in E$ such that

$$\varphi(q) = \min_{u \in E} \varphi(u)$$

Hence the set
$A^* := \{y \in E : \varphi(y) = \min_{u\in E} \varphi(u)\} \neq \emptyset$
Since by lemma (0.2) $\lim_{n\longrightarrow\infty} ||x_{s_n,t} - Tx_{s_n,t}|| = 0$ implies $\lim_{n\longrightarrow\infty} ||x_{s_n,t} - Ax_{s_n,t}|| = 0$ and fix(T)= Fix(A), then for $v \in A^*$ we have
$\varphi(Av) = \mu_n||x_{s_n,t} - Av||^2 = \mu_n||x_{s_n,t} - Ax_{s_n,t} + Ax_{s_n,t} - Av||^2$

$$\leq \mu_n||x_{s_n,t} - Ax_{s_n,t}||^2 + \mu_n||Ax_{s_n,t} - Av||^2$$
$$\leq \mu_n||x_{s_n,t} - v||^2 = \varphi(v)$$

This implies that $\varphi(Av) \leq \varphi(v)$ and so $Av \in A^*$
Now, let $z \in fix(A)$, then $z = Az$, since A^* is closed and convex set, there exist a unique $v^* \in A^*$ such that

$$||z - v^*|| = \min_{u\in A^*} ||z - u||$$

But

$$||z - Av^*|| = ||Az - Av^*|| \leq ||z - v^*||$$

which implies $Av^* = v^*$ and so $A^* \cap Fix(T) \neq \emptyset$
Let $x_t \in A^* \cap Fix(T)$ and $t \in (0,1)$, then it follows that

$$\varphi(x_t) \leq \varphi(x_t + \xi(tf(x_t) + (1-t)Sx_t - x_t))$$

by lemma (0.4) we have

$$\mu_n||x_{s_n,t} - x_t - \xi(tf(x_t) + (1-t)Sx_t - x_t)||^2 \leq \mu_n||x_{s_n,t} - x_t||^2$$
$$-2\mu_n\langle\xi(tf(x_t) + (1-t)Sx_t - x_t), j(x_{sn,t} - x_t - \xi(tf(x_t) + (1-t)Sx_t - x_t)\rangle$$

which implies that

$$\mu_n\langle(tf(x_t) + (1-t)Sx_t - x_t), j(x_{s_n,t} - x_t - \xi(tf(x_t) + (1-t)Sx_t - x_t)\rangle \leq 0$$

Moreover,
$\mu_n\langle tf(x_t) + (1-t)Sx_t - x_t, j(x_{s_n,t} - x_t)\rangle = \mu_n\langle tf(x_t) + (1-t)Sx_t - x_t, j(x_{s_n,t} - x_t)$

$$-j(x_{s_n,t} - x_t - \xi(tf(x_t) + (1-t)Sx_t - x_t))\rangle$$
$$+\mu_n\langle(tf(x_t) + (1-t)Sx_t - x_t), j(x_{s_n,t} - x_t - \xi(tf(x_t) + (1-t)Sx_t - x_t)\rangle$$
$$\leq \mu_n\langle tf(x_t) + (1-t)Sx_t - x_t, j(x_{s_n,t} - x_t)$$
$$-j(x_{s_n,t} - x_t - \xi(tf(x_t) + (1-t)Sx_t - x_t))\rangle$$

Since j is $norm - to - weak^*$ uniformly continous on bounded subsets of E, then as $\xi \to 0$ we have
$\mu_n\langle tf(x_t) + (1-t)Sx_t - x_t, j(x_{s_n,t} - x_t)\rangle \leq 0$

Now, using (8) again we have

$$\begin{aligned}||x_{s_n,t}-x_t||^2 = & st\langle f(x_{s_n,t})-f(x_t), j(x_{s_n,t}-x_t)\rangle \\ & +s(1-t)\langle Sx_{s_n,t}-Sx_t, j(x_{s_n,t}-x_t)\rangle \\ & +(1-s)\langle Tx_{s_n,t}-Tx_t, j(x_{s_n,t}-x_t)\rangle \\ & +st\langle f(x_t)-x_t, j(x_{s_n,t}-x_t)\rangle \\ & +s(1-t)\langle Sx_t-x_t, j(x_{s_n,t}-x_t)\rangle \\ & \leq (1-st(1-\beta))||x_{s_n,t}-x_t||^2 \\ & +st\langle f(x_t)-x_t, j(x_{s_n,t}-x_t)\rangle \\ & +s(1-t)\langle Sx_t-x_t, j(x_{s_n,t}-x_t)\rangle\end{aligned}$$

It turns out that

$$||x_{s_n,t}-x_t||^2 \leq \frac{1}{t(1-\beta)}\langle tf(x_t)+(1-t)Sx_t-x_t, j(x_{s_n,t}-x_t)\rangle \quad \forall x_t \in Fix(T) \tag{10}$$

and so,

$$\mu_n||x_{s_n,t}-x_t||^2 \leq 0$$

Thus, there exist a subnet $\{x_{s_n,t}\}$ of the net $\{x_{s,t}\}$ such that

$$\lim_{n\longrightarrow\infty} x_{s_n,t} = x_t$$

Letting $n \longrightarrow \infty$ in (10) and putting $x_t = z$ we have

$$||x_t-z||^2 \leq \frac{1}{t(1-\beta)}\langle tf(z)+(1-t)Sz-z, j(x_t-z)\rangle \quad \forall z \in Fix(T)$$

so, x_t is a solution of the following variational inequality

$$x_t \in Fix(T), \langle tf(z)+(1-t)Sz-z, j(x_t-z)\rangle \geq 0, \quad \forall z \in Fix(T)$$

By lemma (0.3), when $C = Fix(T), F = t(I-f)+(1-t)(I-S)$ we obtain the equivalent dual variational inequality:

$$x_t \in Fix(T), \langle tf(x_t)+(1-t)Sx_t-x_t, j(x_t-z)\rangle \geq 0, \quad \forall z \in Fix(T) \tag{11}$$

Next, we proof that for each $t \in (0,1)$, as $s \to 0$ $\{x_{s,t}\}$ converges in norm to $x_t \in Fix(T)$. Assume $x_{s_n',t} \to x_t'$ as $s_n' \to 0$, Similar to the above proof, we have $x_t' \in Fix(T)$ which solves the following variational inequality:

$$x_t' \in Fix(T), \langle tf(x_t')+(1-t)Sx_t'-x_t', j(x_t'-z)\rangle \geq 0, \quad \forall z \in Fix(T) \tag{12}$$

Taking $z = x_t'$ in (11) and $z = x_t$ in (12), we have

$$\langle tf(x_t)+(1-t)Sx_t-x_t, j(x_t-x_t')\rangle \geq 0 \tag{13}$$

$$\langle tf(x_t')+(1-t)Sx_t'-x_t', j(x_t'-x_t)\rangle \geq 0 \tag{14}$$

Adding (13) and (14), and since f is strongly pseudocontractive and S is pseudocontractive, we have

$$0 \leq t\langle (I-f)x_t-(I-f)x_t', j(x_t'-x_t)\rangle + (1-t)\langle (I-S)x_t-(I-S)x_t', j(x_t'-x_t)\rangle$$

$$\leq -t(1-\beta)||x_t'-x_t||^2$$

which implies that $x_t' = x_t$. Hence the net $\{x_{s,t}\}$ converges in norm to $x_t \in Fix(T)$ as $s \to 0$.
Now, we show that x_t is bounded.
Since $\Omega \subset Fix(T)$, for any $y \in \Omega$ taking $z = y$ in (11) we obtain

$$\langle tf(x_t)+(1-t)Sx_t-x_t, j(x_t-y)\rangle \geq 0 \tag{15}$$

Since I - S is accretive, for any $y \in \Omega$, we have

$$\langle Sx_t-x_t, j(x_t-y)\rangle \leq \langle Sy-y, j(x_t-y)\rangle \leq 0 \tag{16}$$

combining (15) and (16) we have

$$\langle f(x_t)-x_t, j(x_t-y)\rangle \geq 0, \forall y \in \Omega \tag{17}$$

i.e,

$$\langle f(x_t)-y+y-x_t, j(x_t-y)\rangle \geq 0, \forall y \in \Omega$$

Hence, $||x_t - y||^2 \leq \langle f(x_t) - x_t, j(x_t - y)\rangle$

$$= \langle f(x_t) - f(y), j(x_t - y)\rangle + f(y) - y, j(x_t - y)\rangle$$
$$\leq \beta||x_t - y||^2 + \langle f(x_t) - x_t, j(x_t - y)\rangle$$

Hence,

$$||x_t - y||^2 \leq \frac{1}{1-\beta}\langle f(y) - y, j(x_t - y)\rangle \quad (18)$$

which implies that

$$||x_t - y||^2 \leq \frac{1}{1-\beta}||f(y) - y||.$$

Thus, $\{x_t\}$ is bounded.
Finally, we show that $x_t \to x^* \in \Omega$ which is a solution of variational inequality (4).
Since f is strongly pseudocontractive, it is easy to see that the solution of the variational inequality (4) is unique.
Next, we prove that $\omega_\omega(x_t) \subset \Omega$;, i.e, if (t_n) is a null sequence in (0,1) such that $x_{t_n} \rightharpoonup x^{'}$ as $n \to \infty$, then $x^{'} \in \Omega$. Indeed, it follows from (11) that

$$\langle (I - S)x_t, j(z - x_t)\rangle \geq \frac{t}{1-t}\langle (I - f)x_t, j(z - x_t)\rangle$$

Since I-S is accretive, from the above inequality, we obtain

$$\langle (I - S)z, j(z - x_t)\rangle \geq \frac{t}{1-t}\langle (I - f)x_t, j(z - x_t)\rangle, \forall z \in Fix(T). \quad (19)$$

Letting $t = t_n \to 0$ in (19), we have

$$\langle (I - S)z, j(z - x_t)\rangle \geq 0, \forall z \in Fix(T)$$

which is equivalent to its dual variational inequality by lemma (0.3)

$$\langle (I - S)x^{'}, j(z - x^{'})\rangle \geq 0, \forall z \in Fix(T)$$

Since Fix(T) is closed and convex, then it is weakly closed. Thus, $x^{'} \in Fix(T)$ by virtue of $x_t \in Fix(T)$. So, $x^{'} \in \Omega$.
Lastly, we show that $x^{'} = x^*$, the unique solution of (4). In fact, taking $t = t_n$ and $y = x^{'}$ in (18), we obtain

$$||x_{t_n} - x^{'}||^2 \leq \frac{1}{1-\beta}\langle f(x^{'}) - x^{'}, j(x_{t_n} - x^{'})\rangle.$$

which together with $x_{t_n} \rightharpoonup x^{'}$ implies that $x_{t_n} \to x^{'}$ as $t_n \to 0$.Let $t = t_n \to 0$ in (17), we have

$$f(x^{'}) - x^{'}, j(x^{'} - y)\rangle \geq 0, \forall y \in \Omega. \quad (20)$$

Thus, it follows from (20) and $x^{'} \in \Omega$ that $x^{'}$ is a solution of Variational inequality (4). By uniqueness, we have $x^{'} = x^*$.
Therefore,$x_t \to x^*$ as $t \to 0$.
The proof is completes. □

The following corollary follows directly from theorem (0.5)

Corollary 0.6. *Let E be a real reflexive Banach space that admit a weakly sequencially continous duallity mapping from E to E^*. Let C be a nonempty closed convex subset of E. Let $f : C \to C$ be a Lipschitz strongly pseudocontraction, $S : C \to C$ be a Lipschitz pseudocontraction and $T : C \to C$ be a continous pseudocontraction with $Fix(T) \neq \emptyset$. Suppose that the solution set Ω of the VI(TS, C) is nonempty. Let for each $(s,t) \in (0,1)^2$, $\{x_{s,t}\}$ be defined by (8). Then for each fixed $t \in (0,1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a point $x_t \in Fix(T)$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution x^* of the following variational inequality*

$$x^* \in \Omega, \langle (I - f)x^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in \Omega. \quad (21)$$

Hence, for each null sequence $\{t_n\} \in (0,1)$, there exist another null sequence $\{s_n\} \in (0,1)$, such that the squence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

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