Construction of *p*-Adic Gibbs Measure for *p*-Adic λ-Ising Model

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Abstract: In this work we establish the existence of the p-adic Gibbs measure for the p-adic λ-Ising model on the Cayley tree of order two. In the previous studies, p-adic Ising model and p-adic λ-model were distinctly studied in many papers with the various properties. In this work, it is the first we combined both p-adic Ising model and p-adic λ-model on the Cayley tree of order two. In this research we only establish the model and we proved the existence of the p-adic Gibbs measures in the p-adic case. Here, we use the methods of p-adic analysis, and then, our results do not work in the real case.

Keywords: p-Adic Numbers, p-Adic Quasi Gibbs Measure, Dynamical System, Cayley Tree

1. Introduction

In this paper, we continue the study of p-adic Ising model and p-adic λ-model which was started in (Vannimenus, 1981) and (Khamraev et al., 2004). In these models spins take values $\{\pm 1\}$. In the mentioned works the authors had studied p-adic Ising model and p-adic λ-model separately. And they investigated uniqueness of the Gibbs measures for the models on the Cayley tree. In this work we continued the investigation on the uniqueness of the p-adic Gibbs measures for the model of the combination of these two models. This approach makes our work different from the previous studies. Recently, in the papers of (Mukhamedov, 2012), and (Mukhamedov, 2013), various aspects of these models were studied such that "notions of phase transition: *phase transition* and *quasi phase transition*." and many other properties of these models. In this research we only investigated the essential necessity of such kind of measures which is satisfaction of the Kolmogorov's Consistency condition. Furthermore, such kind of models and analysis of these models provides the solution of many statistical physics models which are not represented usual probability theory.

Organization of the paper is first we gave some definitions, lemmas and theorems in the preliminary section and then we proposed our main theorem in the third section and we proved the p-adic Quasi Gibbs measures occurs for the p-adic λ-Ising model on the Cayley tree of order two.

2. Preliminaries

2.1 *p-adic numbers:* In what follows p is a fixed prime number. The set $\mathbb{Q}_p$ is a completion of the rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p: \mathbb{Q} \to \mathbb{R}$ given by

$$|x|_p = \begin{cases} p^{-r} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases} \tag{1}$$

where $x = p^r \frac{m}{n}$, $\forall x \in \mathbb{Q}$ and $m \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ s.t. $(p, m) = (p, n) = 1$. The absolute value $|\cdot|_p$ is named non-Archimedean and the most important and useful property of this absolute value is satisfying the following inequality which is called "strong triangle inequality."

$$|x + y|_p \leq \max\{|x|_p, |y|_p\} \tag{2}$$

i.e. if $|x|_p > |y|_p$ then $|x + y|_p = |x|_p$. This property is the most crucial property of non-Archimedean metric. Any p-adic number $x \in \mathbb{Q}_p$, where $x \neq 0$ is uniquely represented in the form

$$x = p^r(x_0 + x_1 p^1 + x_2 p^2 + \cdots) \tag{3}$$

where $r \in \mathbb{Z}$ and x_i are integers, $0 \leq x_i < p$, $x_0 > 0, i = 0,1,2, \ldots$. This form is called the canonic form of $x \in \mathbb{Q}_p$ and $|x|_p = p^{-r}$.

2.2. p-adic integers: Let $x \in \mathbb{Q}_p$ be a p-adic number. Then the following set is called the p-adic integers.

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \tag{4}$$

For each $c \in \mathbb{Q}_p$, and $r > 0$, $B(c, r) = \{x \in \mathbb{Q}_p : |x - c|_p \leq r\}$ is called the p-adic ball. Remind that p-adic exponential function is defined by

$$exp_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \tag{5}$$

and this function is convergent for every $x \in B(0, p^{-\frac{1}{p-1}})$. In (Koblitz, 1977) it was proved that

$$\left|exp_p(x)\right|_p = 1, \qquad \left|exp_p(x) - 1\right|_p < 1 \tag{6}$$

In generally we do our investigation in the following set;

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p : |x|_p = 1 \text{ and } |x - 1|_p < p^{-\frac{1}{p-1}}\right\}. \tag{7}$$

2.3. p-adic measure: Suppose that $(X, \mathcal{B})$ is a measurable space where $\mathcal{B}$ is an algebra of subsets of X. A function $\mu: \mathcal{B} \to \mathbb{Q}_p$ is called p-adic measure if the following equality holds;

$$\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i). \tag{8}$$

If $\mu(X) = 1$ then the p-adic measure is called "*probability measure*."

2.4. Cayley Tree: A Cayley tree Γ^k of order $k \geq 1$ is an semi-infinite tree, i.e., a graph without cycles with exactly $k+1$ edges issuing from each vertex except $x^{(0)}$. Let us represent the Cayley tree as $\Gamma^k = (V, \Lambda)$, where V is the set of all vertices of Γ^k, Λ is the set of all edges of Γ^k. Any two vertices x and y, $x, y \in V$ are said to be *nearest-neighbors* if $d(x, y) = 1$ and it is shown by

$l = < x, y >$. And two vertices $x, y \in V$ are said to be *the next nearest neighbors* if $d(x, y) = 2$. Here there exists two possibilities which are one is one level *next nearest neighbors* and the other is prolonged *next nearest neighbors* The distance $d(x, y), x, y \in V$ is the shortest path between any two vertices on the Cayley tree Γ^k. With a fixed vertex $x^0 \in V$, the following set denotes the set of vertices on the same level

$$W_n = \{x \in V \mid d(x, x^0) = n\},$$

And the following set represents the set of all vertices on the Γ^k;

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}.$$

and L_n is the set of all edges in V_n. The set of direct successors S(x) is defined as;

$$S(x) = \{ y \in W_n : d(x, y) = 1, x \in W_{n-1}\}.$$

3. Constraction of p-Adic Gibbs Measure

In this section we consider the mixed type p-adic λ-Ising model where spins $\sigma(x)$ take values in the state space $\Phi = \{-1, +1\}$ where are assigned to the vertices of the Cayley tree $\Gamma^k = (V, L)$. A configuration σ on V is defined as a function s.t. $x \in V \rightarrow \sigma(x) \in \Phi$. By using the similar sense the configurations σ_n and w are defined on V_n and W_n, respectively. $\Omega = \Phi^V$ coincides with the set of all configuration on V. It is easy to see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. With using this we can get the concatenation of $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $w \in \Omega_{W_n}$ as follows;

$$(\sigma_{n-1} \vee w) = \begin{cases} \sigma_{n-1}(x), \text{if } x \in V_{n-1}, \\ w(x), \text{if } x \in W_n . \end{cases}$$

It is obvious that $\sigma_{n-1} \vee w \in \Omega_{V_n}$.

Let $\lambda: \Phi \times \Phi \rightarrow \mathbb{Z}$ be a function for each edge $< x, y > \in L$. Then the Hamiltonian of the mixed type p-adic λ-Ising model $H_n: \Omega_{V_n} \rightarrow \mathbb{Z}$ is defined by

$$H_n(\sigma) = \sum_{<x,y>\in L_n} \lambda(\sigma(x), \sigma(y)) + J \sum_{>x,y<\in W_n} \sigma(x)\sigma(y) \tag{9}$$

Where first part of the sum depends on the all nearest neighbors, $< x, y >$ and second part depends on all on level next nearest neighbors, $> \overline{x, y} <$.

Let $\rho \in \mathbb{Q}_p$ and $\boldsymbol{h}: x \in V - \{x^{(0)}\} \rightarrow h_x \in \mathbb{Q}_p$ be a mapping. For any $n \in \mathbb{N}$, we define the p-adic probability measure, $\mu_h^{(n)}$ due to model on Ω_{V_n} as follow;

$$\mu_{h,\rho}^{(n)}(\sigma) = \frac{1}{Z_{n,\rho}^{(h)}} \rho^{H_n(\sigma)} \prod_{x\in W_n} (h_x)^{\sigma(x)} \tag{10}$$

where $\sigma \in \Omega_{V_n}$ and $Z_{n,\rho}^{(h)}$ is called the partition function defined as follows

$$Z_{n,\rho}^{(h)} = \sum_{\sigma\in \Omega_{V_n}} \rho^{H_n(\sigma)} \prod_{x\in W_n} (h_x)^{\sigma(x)}. \tag{11}$$

Well-known Kolmogorov's extension theorem (Shiryaev, 1980) provides the one of the central results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions. Remember that a p-adic probability measure μ on Ω is compatible if $\mu_h^{(n)}$ holds the following equality;

$$\mu\left(\sigma \in \Omega: \sigma|_{V_n} = \sigma_n\right) = \mu_{h,\rho}^{(n)}(\sigma_n) \text{ for all } \sigma_n \in \Omega_{V_n}, n \in \mathbb{N} \tag{12}$$

The existence of such kind of measures μ is certified by Kolmogorov's extension theorem, (Ganikhodjaev, Mukhamedov & Rozikov, 1998), (Khrennikov & Ludkovsky, 2003) if the $\mu_h^{(n)}$ satisfies the following equation;

$$\sum_{w \in \Omega_{V_n}} \mu_{h,\rho}^{(n)}(\sigma_{n-1} \vee w) = \mu_{h,\rho}^{(n-1)}(\sigma_{n-1}) \tag{13}$$

for all $\sigma_{n-1} \in \Omega_{V_{n-1}}$. This equality guarantees the uniqueness of measure μ on Ω with (12).

With the similar processes in (Mukhamedov, 2012), if the measure $\mu_{h,\rho}^{(n)}$ satisfies the compatibility condition (13) for some function **h** then there exists a unique p-adic probability measure, which we denote by $\mu_{h,\rho}^{(n)}$, since it relates on **h** and ρ. Such a measure $\mu_{h,\rho}^{(n)}$ is called "a generalized p-adic quasi Gibbs measure." corresponding to the p-adic λ- Ising model. By QG(H) we stated the set of all generalized p-adic quasi Gibbs measure associated with functions $\boldsymbol{h} = \{h_x, x \in V\}$.

As a main result of this work, we are going to state the Kolmogorov's Consistency Condition (13) as a main theorem for the new p-adic λ- Ising model and then we prove that condition.

Theorem 3.1. *The measure $\mu_{h,\rho}^{(n)}, n = 1,2,3, \ldots$ satisfies the Kolmogorov's Consistency Condition (13) if and only if for any $x \in V$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} \mathrm{F}\left(h_y, \lambda, J\right) \tag{14}$$

where $S(x)$ is the set of all direct successors of $x \in V$ and

$$\mathrm{F}\left(h_y, \lambda, J\right) = \frac{1}{2}\log\left(\frac{\exp_p\left(2\lambda(1,1)+4h_y+2J\right)+2\exp_p\left(\lambda(1,-1)+\lambda(1,1)+2h_y\right)+\exp_p\left(2\lambda(1,-1)+2J\right)}{\exp_p\left(2\lambda(-1,1)+4h_y+2J\right)+2\exp_p\left(\lambda(-1,1)+\lambda(-1,-1)+2h_y\right)+\exp_p\left(2\lambda(-1,-1)+2J\right)}\right) \tag{15}$$

Proof. Necessity: Suppose that equation (13) holds, we goal to prove (15). To realize this we substitude the equation (9) in the equation (13) hence for any configuration $\sigma_{n-1} \in \Omega_{V_n}$ we get,

$$\mathrm{Z}_n^{-1} \sum_{\sigma^{(n)}} exp_p\left[\sum_{<x,y> \in L_n} \lambda\left(\sigma(x), \sigma(y)\right) + J \sum_{>x,y< \in W_n} \sigma(x)\sigma(y) + \sum_{x \in W_n} {h_x}^{\sigma(x)}\right]$$
$$= \mathrm{Z}_{n-1}^{-1} \sum_{\sigma^{(n)}} exp_p\left[\sum_{<x,y> \in L_{n-1}} \lambda\left(\sigma(x), \sigma(y)\right) + J \sum_{>x,y< \in W_{n-1}} \sigma(x)\sigma(y) + \sum_{x \in W_{n-1}} {h_x}^{\sigma(x)}\right]$$

From here we get

$$\frac{Z_{n-1}}{Z_n} \sum_{\sigma(n)} exp_p\left[\sum_{x \in W_{n-1}} \sum_{y \in S(x)} \lambda\left(\sigma(x), \sigma(y)\right) + J \sum_{y,z \in S(x)} \sigma(y)\sigma(z) + \sum_{y \in S(x)} {h_y}^{\sigma(y)}\right] = \prod_{x \in W_{n-1}} exp_p\left({h_x}^{\sigma(x)}\right)$$

From the last equality we obtain the following

$$\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{\sigma(y)\in\{\pm1\}} exp_p\left[\sum_{x\in W_{n-1}}\lambda(\sigma(x),\sigma(y))+J\sum_{y,z\in S(x)}\sigma(y)\sigma(z)+\sum_{y\in S(x)}{h_y}^{\sigma(y)}\right]$$

$$=\prod_{x\in W_{n-1}} exp_p(\,{h_x}^{\sigma(x)})$$

In the last equation above if we substitute $\sigma(x)=1$ and $\sigma(x)=-1$, respectively and then we divide first equation to the second one then we can get the required equation:

$$\mathrm{F}(h_y,\lambda,J)=\frac{1}{2}\log\left(\frac{\exp_p(2\lambda(1,1)+4h_y+2J)+2\exp_p(\lambda(1,-1)+\lambda(1,1)+2h_y)+\exp_p(2\lambda(1,-1)+2J)}{\exp_p(2\lambda(-1,1)+4h_y+2J)+2\exp_p(\lambda(-1,1)+\lambda(-1,-1)+2h_y)+\exp_p(2\lambda(-1,-1)+2J)}\right)$$

Sufficiency: Assume that the equation (14) is valid. Then it provides the existence of $a(x)\in\mathbb{Q}_p$
such that;

$$\prod_{y\in S(x)}\sum_{\tilde\sigma(y)\in\{\pm1\}} exp_p\left[\lambda(\sigma(x),\tilde\sigma(y))+J\sum_{z\in S(x),\eta(z)\in\phi}\tilde\sigma(y)\eta(z)+\tilde\sigma(y)h_y\right] \tag{16}$$
$$=a(x)exp_p(\sigma(x)h_x)$$

where $\sigma(x)\in\{\pm1\}$. From the previous equality, one obtains

$$\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{\tilde\sigma(y)\in\{\pm1\}} exp_p\left[\lambda(\sigma(x),\tilde\sigma(y))+J\sum_{z\in S(x),\eta(z)\in\phi}\tilde\sigma(y)\eta(z)+\tilde\sigma(y)h_y\right]$$
$$=\prod_{x\in W_{n-1}} a(x)exp_p(\sigma(x)h_x) \tag{17}$$

Now if we multiply both sides of (17) by $exp_p(H_{n-1}(\sigma))$ and representing

$$A_n(x)=\prod_{x\in W_n} a(x) \tag{18}$$

then we get

$$U_n exp_p(H_{n-1}(\sigma))\prod_{x\in W_{n-2}}\prod_{y\in S(x)}\left(h_{xy,\sigma(x)\sigma(y)}\right)^{\sigma(x)\sigma(y)}$$

$$=exp_p(H_{n-1}(\sigma))\prod_{x\in W_{n-2}}\prod_{y\in S(x)}\sum_{\tilde\sigma(y)\in\{\pm1\}} exp_p\left[\lambda(\sigma(x),\tilde\sigma(y))+J\sum_{z\in S(x),\eta(z)\in\phi}\tilde\sigma(y)\eta(z)+\tilde\sigma(y)h_y\right].$$

From (11) it holds that

$$U_{n-1}Z_{n-1}^{(h)}\mu_h^{(n-1)}=Z_n^{(h)}\sum_\eta \mu_h^{(n)}(\sigma\vee\eta) \tag{19}$$

For each $n\in\mathbb{N}$ the measure $\mu_h^{(n)}$ is a probability measure i.e. $\mu_h^{(n)}(\sigma)=1$. Therefore from (19) we get the required one;

$$U_{n-1}Z_{n-1}^{(h)} = Z_n^{(h)} \quad (20)$$

(19) and (20) imply (13). The proof is completed. This theorem provides that our model has unique p-adic quasi Gibbs measure.

4. Conclusion

In this work we investigated the contraction and uniqueness of the p-adic quasi Gibbs measures for the p-adic λ-Ising model on the Cayley tree of order two. In our main theorem we proved that our model satisfies the Kolmogorov's Consistency theorem which guarantees the uniqueness and existence of such kind of measures.

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