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Solution of First-Order Differential Equation Using Fourth-Order Runge-Kutta Approach and Adams Bashforth Methods

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Abstract: In this research, we investigate the solution of first-order differential equations (DEs) using Runge- Kutta fourth-order method (RKM) and Adams-Bashforth methods (ABMs). In this work we consider fourth-order RKM and ABMs for solving first order DEs. The method proof to be simple, easy, accurate and efficient technique for solving first order DEs. Moreover, there are unlimited application of fourth-order RK4 and ABMs for solving first-order DE in science, engineering, economics, social science, biology and business. These play an important role in science and engineering. Some examples are giving and solved to support the efficiency of our methods which are demonstrated by figures.

Keywords: Ordinary Differential Equation, Runge-kutta fourth-order method, Adam Bashford methods, Numerical Approximation.

1. INTRODUCTION

Almost all systems undergoing change can be described by DEs. They are present everywhere in the fields of engineering, science, economics, social science, business, biology, health care, etc. The nature of these equations has been investigated by several mathematicians for hundreds of years, there are many effective solution methods. A purely analytical solution to the equations is frequently impractical due to the complexity or size of the systems that differential equations represent. Computer simulations and numerical methods are useful in these complex systems. Before programmable computers existed, methods of solving DEs based on numerical estimations were devised. It was typical to see rooms filled with personnel (mostly women) using mechanical calculators to solve DE systems numerically for military calculations during the second world conflict. Before the advent of programmable computers, analog computers used analogs of an electrical system to investigate mechanical, thermal, or chemical processes. With the development of programmable computers, systems of DEs of increasing complexity can be solved using straightforward programs created to run on a standard computer. (Storey, 2004). The ordinary differential equations (ODE) is an important mathematical equation in natural physical processes. Analytical approaches have many difficulties for approximating the majority of ODEs. As a result, being able to find a numerically precise solution is critical (Ray, 2018). One of the important areas of mathematics, particularly when it comes to dealing with scientific (engineering) problems, is differential equation. Differential equations may be used to build mathematical models for a variety of issues that people have faced, particularly in the scientific (technical) area. With a variety of solutions available, techniques are being developed to solve differential equations, taking into account their extensive application. Analytical and numerical techniques they can be used to solve common DEs. The analytical technique produces answers that are typically precise values, whereas the numerical method produces solutions that are approximations of the true value. Since approximation does not need to apply calculus theorems, differential equations solved numerically only generate approximations. The use of one requires precision since numerical resolution incorporates numerous factors. Numerical differential equations started to be resolved as science and technology advanced, particularly in the area of computers (Polla, 2013). There are several numerical integration techniques available to an engineer who wants to integrate a set of ordinary differential equations. Both single-step fourth-order RKM and multi-step ABMs integrators of any order are covered. Additionally, both fixed-step and variable-step variants of these integrators are offered. Notwithstanding this, the fixed-step, RK4 approach is nevertheless employed occasionally even when the issue at hand does not call for fixed-step integration. Engineers sometimes rely on what has worked in the past and may not always have time to experiment with numerical approaches, which may be the cause of this. The fourth-order RKM technique already has step-size control. To estimate the truncation error of the fourth-order procedure, one technique called doubling takes two standard steps and a double step at the same time (D. G. Hull, 1977). The use of Runge-Kutta methods for starting and modifying intervals as well as issues such as stability and reduction are discussed the majority of round off errors aren't serious (T. Hull, Enright, Fellen, & Sedgwick, 1972). The explicit ABMs, which belong to the Adams Family, are the most popular linear multistep techniques for no stiff situations. The Stone Weierstrass Theorem serves as the foundation for the ABMs (Dattani, 2008). Runge-Kutta technique is a one-step numerical method since it only needs one prior point to compute a new value. The Runge-Kutta technique of order four is often utilized. Runge-Kutta procedures come in a variety of forms, and they all depend on the value of n that is utilized (Pagalay, 2016). In the fields of science and engineering, numerical methods are widely used to solve a variety of linear and nonlinear ODEs. Euler approach, Picard method, Taylor sequential method, fourth-order RKM method are only a few of the well-known techniques created to solve ODEs. The advantage of RKM approaches over the methods previously outlined is that they are intended to deliver more precision with the benefit of just requiring the function values at certain specific locations on the sub-interval, which eliminates the need for computations for higher order derivatives (Arora, Joshi, & Garki, 2020).

One of the biggest challenges facing scientists has been solving complex differential equations. In order to address these challenges, researchers have suggested a numerical method for solving partial differential equations (PDEs), fractional differential equations (FDEs), and ordinary differential equations (ODEs) in works by (Ibrahim, 2020; Ibrahim, & Isah, 2021; Isah, & Ibrahim, 2021; Ibrahim, & Isah, 2022; Salisu, 2022b). This means that commutativity is important from a practical point of view.

(Ibrahim, & Koksal, 2021a) examined commutativity and its effect on sensitivity when there are non-zero starting conditions (ICs). Meanwhile, using cascaded pairs of second-order commutative systems, (Salisu, 2022a; Salisu, 2022c) explored the realization and decomposition of fourth-order Linear Time-Varying Systems (LTVSs) with non-zero ICs. (Ibrahim, & Koksal, 2021b; Salisu, & Rababah, 2022) conducted an analogous study. Additionally, a numerical approximation procedure for degree reduction of curves and surfaces was developed by (Ibrahim, 2020; Ibrahim, & Isah, 2021; Isah, & Ibrahim, 2021; Ibrahim, & Isah, 2022; Salisu, 2022b). These methods provide viable ways to deal with intricate PDEs, FDEs, and ODEs.

The above study investigates fourth-order RKM approach and ABMs to solve a first-order DE. This study's goal is to apply recommended approaches to find numerical solutions for first-order DEs.

2. PRELIMINARIES

The Euler technique for solving differential equations of the first and second orders is discussed in this section. We also discuss how to solve partial differential equations (PDEs), fractional differential equations (FDEs), and ordinary differential equations (ODEs) using the Taylor polynomial formula, the Newton method, and the Cubic Spline Interpolation method. Although these techniques have proven effective in solving ODEs numerically, the main purpose of this work is to solve first-order ODEs using the fourth-order RKM technique and ABM approaches, as will be explained in the following section.

2.1 Euler method

Within the realm of mathematics and Computational Science, Euler's method, also known as forward

Euler's method, serves as a numerical technique of first-order for solution ODEs when provided with a specific initial value. This approach represents the fundamental explicit technique for numerically integrating ODEs and stands as the most straightforward RKM. Euler's method is attributed to Leonhard Euler, who introduced it in his book "Integration of Institutional Accounts" (published 1768-1870).

The goal of Euler's method is to obtain approximations of a well-placed initial value problem.

$$y'(x) = f(x, y), \tag{1}$$

$$y(x_o) = y_0. \tag{2}$$

Beginning with the initial conditions (ICs) y_0:

$$x_{n+1} = x_n + h, \tag{3}$$

$$y_{n+1} = y_n + hf(x_n, y_n), \tag{4}$$

where h is the step function, x_{n+1} is the independent value and y_{n+1} is the numerical solution.

2.2 Taylor Series

A Taylor polynomial of degree n for a function f at a point $x_0 \in [a, b]$ is a polynomial expression that approximates the function near the point x_0. The polynomial is defined using the coefficients of the function's derivatives evaluated at x_0.

$$Q_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0). \tag{5}$$

The Taylor polynomial (5) has the characteristic that it shares identical derivatives up to order n with the function f when evaluated at $x = x_0$, i.e.

$$Q_n^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0,1, \dots, n,$$

To verify that the derivatives of the Taylor polynomial $Q_n(x)$ match those of the function f, one can perform differentiation. The Taylor polynomial provides a reliable approximation of the function $f(x)$ in the vicinity of x_0. The discrepancy between f and its Taylor polynomial is quantified by the remainder term in Taylor's formula.

$$f(x) - Q_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_o)^{n+1} \tag{6}$$

The expression provided for the remainder involves Lagrange's form, which is useful for calculating error bounds. The interval $[a, b]$ contains the point x, and there exists another point ξ between x and x_0 (when x is not equal to x_0). It is assumed that the derivative $f^{(n+1)}$ is continuous on $[a, b]$, and therefore it is bounded within this interval.

$$M_{n+1} = \max_{[a,b]} |f^{(n+1)}(x)| < +\infty. \tag{7}$$

On the basis of (6), we have

$$|f(x) - Q_n(x)| \le \frac{M_{n+1}}{(n+1)!} |x - x_o|^{n+1},$$

$$\max_{[a,b]} |f(x) - Q_n(x)| \le \frac{M_{n+1}}{(n+1)!} l^{n+1}, \tag{8}$$

where $l = \max[x_0 - a, b - x_0]$.

The equation (7) indicates that the error resulting from the approximation of function $f(x)$ by Taylor polynomial (5) can be expressed as $O(|x - x_o|^{n+1})$, while equation (8) is used to provide an estimation of the maximum error across the entire interval $[a, b]$.

2.3 Newton's Method

Newton's method which is also known as Newton Raphson's method, is an algorithm that is used to find roots. It involves using the initial the Taylor series terms of the function $f(x)$ are close to a potential root. That it essential to note that although this method is occasionally referred to as Newton's iteration, in this context the last statement is reserved for its application in calculating square roots.

Taylor series of f(x) around this point $x = x_0 + \varepsilon$ given before

$$f(x_0 + \epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{1}{2} f''(x_0)\varepsilon^2 + \cdots \tag{9}$$

Retaining first-class conditions only,

$$f(x_0 + \epsilon) \approx f(x_0) + f'(x_0)\epsilon. \tag{10}$$

Equation (9) is the equation of the tangent line to the curve at $(x_0, f(x_o))$, so $(x_1, 0)$ is the place where this tangent line intersects x_axis.

Setting $f(x_0 + \varepsilon) = 0$ and solving (10) for $\epsilon \equiv \epsilon_0$ gives

$$\epsilon_0 = -\frac{f(x_0)}{f'(x_0)},$$

It is the first modification of the root position. Empowered $x_1 = x_0 + \epsilon_0$, by calculating a new ϵ_1, using

$$\epsilon_n = -\frac{f(x_n)}{f'(x_n)}.$$

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Unfortunately, this procedure can be unstable near a horizontal asymptote or local extremes. However, with a good initial selection of the root position, the algorithm can be applied iteratively to obtain it

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (11)$$

for $n = 1,2,3,\dots$ the initial point x_0 which provides a safe approximation to Newton's method is called the approximate zero (Weisstein., E , 2002).

2.4 Cubic Spline Interpolation Polynomial

Cubic spline interpolation is a form of spline interpolation frequently utilized to mitigate the issue of Runge's phenomenon. It represents a specific instance of spline interpolation. The approach yields an interpolator polynomial that is smoother and possesses lower error than alternative interpolating polynomials, such as Lagrange and Newton polynomials (Ibrahim, 2022c).

Given a set of $n+1$ data points (x_i, y_i) where no two x_i are the same and $a = x_0 < x_1 < \dots < x_n = b$, the spline $S(x)$ is a function satisfying

1. $s(x) \epsilon C^2[a,b]$;
2. On each subinterval $[x_{i-1}, x_i]$, $S(x)$ is a polynomial of degree *3*, where $i = 1, \dots, N$

$S(x) = y_i$, for all $i = 0,1,\dots,n$.

3. THE FOURTH-ORDER RUNGE-KUTTA APPROACH AND ADAMS BASHFORTH METHODS

In this section. We are going to consider the fourth-order RKM and ABMs.

3.1 Adam Bashforth method

The Adams Bashforth techniques are considered to be explicit methods, where the selection of coefficients $a_{s-1} = -1$and $a_{s-2} = \cdots = a_0$ =0, while the b_j are determined by the desired order s of the method. This characteristic feature ensures the uniqueness of the methods. Specifically, the Adams Bashforth methods with orders s = 1, 2, 3, 4, and 5 are examples of such explicit methods.

$y_{i+1} = y_i + hf(t_i, y_i)$ (This is the Euler method)

$$y_{i+2} = y_{n+1} + h\left(\frac{3}{2}f(t_{n+1}, y_{n+1}) - \frac{1}{2}f(t_n, y_n)\right), \quad (12)$$

$$y_{n+3} = y_{n+2} + h\left(\frac{3}{2}f(t_{n+2}, y_{n+2}) - \frac{16}{12}f(t_{n+1}, y_{n+1}) + \frac{5}{12}f(t_n, y_n)\right), \quad (13)$$

$$y_{n+4} = y_{n+3} + h\left(\frac{55}{24}f(t_{n+3}, y_{n+3}) - \frac{59}{24}f(t_{n+2}, y_{n+2}) + \frac{37}{24}f(t_{n+1}, y_{n+1}) - \frac{9}{24}f(t_n, y_n)\right), \quad (14)$$

$$y_{n+5} = y_{n+4} + h(\frac{1901}{720}f\left(t_{n+4}, y_{n+4} - \frac{2774}{720}f(t_{n+3}, y_{n+3})\right) + \frac{2616}{720}f(t_{n+2}, y_{n+2}) - \frac{1274}{720}f(t_{n+1}, y_{n+1}) + \frac{251}{720}f(t_n, y_n). \quad (15)$$

To obtain the coefficients b_j one can follow these steps: employ polynomial interpolation to determine the polynomial p of degree $s-1$, and subsequently extract the coefficients from it.

$$p(t_{n+i}) = f(t_{n+i,} y_{n+i}), \text{ for } i = 0,1,\dots..s-1.$$

When presented with an ODE in the form of $\frac{dy}{dx} = f(x,y)$, along with an IC of $y(x_0) = y_0$, the goal is to figure out how to solve *y* in terms of *x*. This is frequently achieved by an integration procedure that entails splitting the variables out of the equation, integrating each side with regard to its own variable, and then utilizing the initial condition to solve for y. The final solution can then be presented in a way that makes sense and can be used in particular situations. To locate (x_n). We can use a variety of numerical techniques, including the Taylor series, the Modified Euler method, the Euler method, Picard's approach, and the Runge Kutta method, to determine the values of $y(x_1)$, $y(x_2)$, and $y(x_3)$ for a given function.

Then calculate, $f_0 = f(x_0, y_0)$

$$f_1 = f(x_1, y_1)$$
$$f_2 = f(x_2, y_2)$$
$$f_3 = (x_3, y_3).$$

By Adams - Bashforth predictor formula:

$$y_4 = y_3 + \frac{h}{24}(-9f_0 + 37f_1 - 59f_2 + 55f_3). \quad (16)$$

Then find $f_4 = f(x_4, y_4)$ since, $x_4 = (x_3 + h)$.

And by Adams - Bashforth corrector formula

$$y_4 = y_3 + \frac{h}{24}(f_1 - 5f_2 + 19f_3 + 9f_4). \quad (17)$$

Recalculating and inserting the result into the AB corrector formula will yield a more precise answer

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for f_4. Iteratively repeating this process is possible until y_4 reaches a stable value and no more changes are noticed (Kavitha, P., & Prathiba, K.).

Runge-Kutta and Adam-mellitus are categorized as multi-step linear methods. This means that the linear set of the previous values y_i and the accompanying function evaluations $f(x_i, y_i)$ across the previous s steps determine the subsequent value of y_{n+1}.

The two-step Runge-kutta method approach was selected due to the simplicity of its equations and ease of implementation. However, it is possible to expand the code to incorporate more intricate Runge-kutta method techniques. Additionally, an algorithm was created to address differential equations using the Runge-kutta method equations and adjusting the size of the steps based on the variance between prediction and correction (Pletinckx, et al, 2017).

3.2 Runge-Kutta Method Fourth-Order

The Runge-Kutta methods are techniques do not need computing or evaluating the derivatives of f, but they do it has a local high level truncation error for Taylor method (t, y). We must first take into account Taylor's Theorem in Two Variables before explaining the concepts behind their derivation. The Taylor techniques described in the previous section, you have the useful quality of higher-order local truncation error, but have the unfavorable requirement of computing and evaluating the derivatives of $f(t, y)$. The Taylor techniques are rarely employed in practice since it is a difficult and time-consuming process for the majority of issues (Burden, Faires, & Burden, 2015).

Let the initial value problem be defined as follows:

$$\frac{dy}{dx} = f(t, y), \quad y(t_o) = y_o. \tag{18}$$

Now we choose step size $h > 0$ and define:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)h, \tag{19}$$

$$t_{n+1} = t_n + h \tag{20}$$

For $n = 0, 1, 2, 3, \ldots$, using

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \tag{21}$$

$$k_4 = hf(x_n + h, y_n + k_3).$$

- k_1 is the slope at the beginning of the period using y (Euler's method);
- k_2 is the slope at the midpoint of the period with y and k_1;
- k_3 is again the slope at the midpoint, but now uses y and k_2;
- k_4 is the slope at the end of the period using y and k_3.

4. APPLICATION

In this section, we make use of the formula and conditions obtained from the previous section and illustrate the numerical solution of first-order differential equations using fourth-order RKM and ABMs.

Example 1: In this example, we want to apply the ABMs by considering the following first-order DEs

$$y' = \frac{ty}{1+t^2}, \quad y(0) = 1, \tag{22}$$

on the interval $0 \le t \le 1$.

The exact solution is given by

$$y_{exact} = \sqrt{1+t^2}. \tag{23}$$

By considering the fourth-order RKM in Eq. (21) with $t_0 = 0$, $y_0 = 1$; $h = 0.1$, we obtain

$$k_1 = f(t_0, y_0) = f(0, 1) = \frac{(0\times1)}{1+0^2} = 0$$

$$k_2 = f\left(t_0 + \frac{h}{2}, y_0 + h\frac{k_1}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + (0.1)\frac{0}{2}\right) = f(0.05, 1) = \frac{(0.05\times1)}{1+(0.05)^2} = 0.049$$

$$k_3 = f\left(t_0 + \frac{h}{2}, y_0 + h\frac{k_2}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + (0.1)\frac{0.049}{2}\right) = f(0.05, 1.00245) = \frac{(0.05\times1.00245)}{1+(0.05)^2} = 0.0499$$

$$k_4 = f(t_0 + h, y_0 + hk_3) = f(0 + 0.1, 1 + (0.1)(0.0499)) = f(0.1, 1.00499) = \frac{(0.1\times1.00499)}{1+(0.1)^2} = 0.099.$$

Then we proceed to find y_1, y_2, y_3 and y_4 with

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \text{ for } n = 0,1,2,3,4, \ldots$$

$$\text{IF } n = 0 \to y_1 = 1 + \frac{0.1}{6}(0 + 2(0.049) + 2(0.0499) + 0.099) = 1.0049$$

$$\text{If } n = 1 \to y_{1+1} = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

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$$y_2 = 1.0049 + \frac{0.1}{6}(0 + 2(0.049) + 2(0.0499) + 0.099) = 1.0096$$

If $n = 2 \rightarrow y_{2+1} = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_3 = 1.0096 + \frac{0.1}{6}(0 + 2(0.049) + 2(0.0499) + 0.099) = 1.0143$$

If $n = 3 \rightarrow y_{3+1} = y_3 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_4 = 1.0143 + \frac{0.1}{6}(0 + 2(0.049) + 2(0.0499) + 0.099) = 1.019$$

Now we have $t_0 = 0,\ t_1 = 0.1,\ t_2 = 0.2,\ t_3 = 0.3,\ t_4 = 0.4$

And $y_0 = 1,\ y_1 = 1.0049,\ y_2 = 1.0096,\ y_3 = 1.0143,\ y_4 = 1.019$

$$y' = \frac{ty}{1+t^2} \rightarrow y'_0 = f(x_0, y_0) = f(0,1) = \frac{(0\times 1)}{1+0^2} = 0$$

$$y'_1 = f(t_1, y_1) = f(0.1, 1.0049) = \frac{(0.1 \times 1.0049)}{1 + (0.1)^2} = 0.099$$

$$y'_2 = f(t_2, y_2) = f(0.2, 1.0096) = \frac{(0.2 \times 1.0096)}{1 + (0.2)^2} = 0.194$$

$$y'_3 = f(t_3, y_3) = f(0.3, 1.0143) = \frac{(0.3 \times 1.0143)}{1 + (0.3)^2} = 0.279$$

$$y'_4 = f(t_4, y_4) = f(0.4, 1.019) = \frac{(0.4 \times 1.019)}{1 + (0.4)^2} = 0.351.$$

Observed that the ABMs have two formula:

1. The Adam Bashforth Predictor method

$$y_{n+1,p} = y_n + \frac{h}{24}\left(-9y'_{n-3} + 37y'_{n-2} - 59y'_{n-1} + 55y'_n\right).$$

If we putting $n = 3$, we get

$$y_{4,p} = y_3 + \frac{h}{24}\left(-9y'_0 + 37y'_1 - 59y'_2 + 55y'_3\right).$$

$$y_{4,p} = 1.0143 + \frac{0.1}{24}(-9(0) + 37(0.099) - 59(0.194) + 55(0.279) = 1.0453$$

Hence, we found approximate solution using Adam Bashforth Predictor method

$$y_{4,p} = 1.0453. \tag{24}$$

2. The Adam Bashforth Corrector method

$$y_{4,c} = y_3 + \frac{h}{24}\left(y'_1 - 5y'_2 + 19y'_3 + 9y'_4\right).$$

$$y_{4,c} = 1.0143 + \frac{0.1}{24}(0.099 - 5(0.194) + 19(0.279) + 9(0.351) = 1.0458.$$

Hence, we found approximate solution using Adam Bashforth Corrector method

$$y_{4,c} = 1.0458. \tag{25}$$

Example 2

In this example, we want to apply the ABMs which consist of predictor and corrected methods.

By considering the following first-order DEs

$$y' = t^2 + ty, \quad y(0) = 1, \tag{26}$$

on the interval $0 \leq x \leq 1$.

The exact solution is given by

$$y_{exact} = \sqrt{\frac{\pi}{2}}\, e^{t^2/2}\, \mathrm{erf}\left[\frac{x}{\sqrt{2}}\right] + e^{t^2/2} - t. \tag{27}$$

By considering the fourth-order RKM in Eq. (21) with $t_0 = 0$, $y_0 = 1$; $h = 0.1$, we obtain

$$k_1 = f(t_0, y_0) = f(0,1) = (0^2 + (0 \times 1) = 0$$

$$k_2 = f\left(t_0 + \frac{h}{2}, y_0 + h\frac{k_1}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + (0.1)\frac{0}{2}\right) = f(0.05,1) = (0.05)^2 + (0.05 \times 1) = 0.0525$$

$$k_3 = f\left(t_0 + \frac{h}{2}, y_0 + h\frac{k_2}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + (0.1)\frac{0.0525}{2}\right) = f(0.05,1.0026) = (0.05)^2 + (0.05 \times 1.0026) = 0.07513$$

$$k_4 = f(t_0 + h, y_0 + hk_3) = f(0 + 0.1, 1 + (0.1)(0.07513) = f(0.1, 1.007513) = (0.1)^2 + (0.1 \times 1.007513) = 0.1107$$

Then we find y_1, y_2, y_3 and y_4

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \text{ for } n = 0,1,2,3,4, \ldots$$

If $n = 0 \rightarrow y_1 = 1 + \frac{0.1}{6}(0 + 2(0.0525) + 2(0.07513) + 0.1107) = 1.0060$

If $n = 1 \rightarrow y_{1+1} = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_2 = 1.0060 + \frac{0.1}{6}(0 + 2(0.0525) + 2(0.07513) + 0.1107) = 1.0120$$

If $n = 2 \rightarrow y_{2+1} = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_3 = 1.0120 + \frac{0.1}{6}(0 + 2(0.0525) + 2(0.07513) + 0.1107) = 1.018$$

If $n = 3 \rightarrow y_{3+1} = y_3 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_4 = 1.018 + \frac{0.1}{6}(0 + 2(0.0525) + 2(0.07513) + 0.1107) = 1.023$$

Now we have $t_0 = 0,\ t_1 = 0.1,\ t_2 = 0.2,\ t_3 = 0.3,\ t_4 0.4$

And $y_0 = 1,\ y_1 = 1.0060,\ y_2 = 1.0120,\ y_3 = 1.018,\ y_4 = 1.023$

$$y' = x^2 + xy \rightarrow y'_0 = f(x_0, y_0) = f(0,1) = (0^2 + (0 \times 1) = 0$$

$$y'_1 = f(x_1, y_1) = f(0.1, 1.0060) = (0.1)^2 + (0.1 \times 1.0060) = 0.1106$$

$$y'_2 = f(x_2, y_2) = f(0.2, 1.0120) = (0.2)^2 + (0.2 \times 1.0120) = 0.2424$$

$$y'_3 = f(x_3, y_3) = f(0.3, 1.018) = (0.3)^2 + (0.3 \times 1.018) = 0.3954$$

$$y'_4 = f(x_4, y_4) = f(0.4, 1.023) = (0.4)^2 + (0.4 \times 1.023) = 0.5692$$

Observed that the ABMs have two formula:

1. The Adams- Bashforth Predictor method

$$y_{n+1,p} = y_n + \frac{h}{24}(-9y'_{n-3} + 37y'_{n-2} - 59y'_{n-1} + 55y'_n)$$

If we putting $n = 3$, we get

$$y_{4,p} = y_3 + \frac{h}{24}(-9y'_0 + 37y'_1 - 59y'_2 + 55y'_3)$$

$$y_{4,p} = 1.018 + \frac{0.1}{24}(-9(0) + 37(0.1106) - 59(0.2424) + 55(0.3954) = 1.0453$$

$$y_{4,p} = 1.0662. \tag{28}$$

2. The Adams- Bashforth Corrector method

$$y_{4,c} = y_3 + \frac{h}{24}(y'_1 - 5y'_2 + 19y'_3 + 9y'_4)$$

$$y_{4,c} = 1.018 + \frac{0.1}{24}(0.1106 - 5(0.2424) + 19(0.3954) + 9(0.5692) = 1.0458$$

$$y_{4,c} = 1.0652. \tag{29}$$

5. CONCLUSION

In conclusion, this study explored the numerical solution of first-order DEs using two popular methods, namely, fourth-order RKM and ABMs. fourth-order RKM is a well-established technique that approaches a solution by iterative evaluation of intermediate values. On the other hand, ABMs is a forecast-corrected scheme that uses past estimates to estimate future values.

[1] It is clear from a thorough investigation and comparison of these two techniques that each has advantages and disadvantages of its own. larger precision and stability are achieved with fourth-order RKM, particularly for tight differential equations, but it comes with a larger computational cost. Conversely, the ABMs are better appropriate for non-extreme problems and are computationally efficient, but they may have stability problems. Overall, this work highlights the advantages and disadvantages of the fourth-order RKM and ABMs and offers insightful information about the numerical solution of first-order differential equations. Additional studies can be conducted to investigate other numerical techniques or examine how these techniques can be applied to multidimensional problems, higher-order differential equations, or particular real-world situations.

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