Development and Isometry of Surfaces Galilean Space G_3

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Received March 3, 2023; Revised May 13, 2023; Accepted June 11, 2023

Cite This Paper in the following Citation Styles

***(a)**: [1] B.M. Sultanov, A. Kurudirek, Sh.Sh. Ismoilov, "Development and Isometry of Surfaces Galilean Space G_3," Mathematics and Statistics, Vol.11, No.6, pp. 965-972, 2023. DOI: 10.13189/ms.2023.110612*

***(b)**: B.M. Sultanov, A. Kurudirek, Sh.Sh. Ismoilov (2023). Development and Isometry of Surfaces Galilean Space G_3. Mathematics and Statistics, 11(6), 965-972. DOI: 10.13189/ms.2023.110612*

Abstract Currently, the study of the geometry of semi-Euclidean spaces is an urgent task of geometry. In the singular parts of pseudo-Euclidean spaces, a geometry associated with a degenerate metric appears. A special case of this geometry is the geometry of Galileo. The basic concepts of the geometry of Galilean space are given in the monograph by A. Artykbaev. Here the differential geometry "in the small" is studied, the first and second fundamental forms of surfaces and geometric characteristics of surfaces are determined. The derivational equations of surfaces, analogs of the Peterson-Codazzi and Gauss formulas are calculated. This paper studies the development and isometry of surfaces in Galilean space. Moreover, the isometry of surfaces in Galilean space is divided into three types: semi-isometry, isometry and completely isometry. This separation is due to the degeneracy of the Galilean space metric. The existence of a development of a surface projecting uniquely onto a plane in general position is proved, as well as the conditions for isometric and completely isometric surfaces of Galilean space. We present the conditions associated with the analog of the Christoffel symbol, providing isometries of the surfaces of Galilean space. An example of isometric, but not completely isometric surfaces in G_3 is given. The concept of surface development for Galilean space is generalized. A development of the surface is obtained, which is uniquely projected onto the plane of the general position. In addition, the Gaussian curvature of the surface has been shown to be completely defined by Christoffel symbols.

Keywords Defects in Curvature, Development of Surface, Galilean Space, Isometry of Surfaces, Uniquely Projected

1 Introduction

The geometry of the Galilean space refers to the geometry of spaces with degenerate metrics. The general theory of spaces with projective metrics with degenerate and non-degenerate metrics is given in the classic monograph "Non-Euclidean Spaces" by B.A. Rozenfeld [1] and O. Roschel [2]. The monograph by A. Artykbaev, D.D. Sokolov is devoted to the study of specific problems of geometry "in the whole" of the Galilean space [3].

After 2000, a broad study of the geometry of the Galilean space began. In this regard, the work of the Professor of the Firat University M.E. Aydin and his students [4, 5], the work of Professor M. Dede from A.Kilis University (Turkey) and his students [6, 7], as well as the work of Professor D.W Yoon [8, 9] from Gyeongsang National University (South Korea).

The theory of the surface of Galilean space is devoted to the works of P. Bansal [10], K. Ilim [11], A. Kazan [12], Z.M. Sipus [13], A. Fatma [14], Z.K. Yuzbasi [15]. In these works, the differential geometry of Galilean space is studied.

In the study of the geometry of non-Euclidean spaces, the superimposed space method is sometimes used, that is, the coordinate system of the non-Euclidean space is considered the coordinate system of the Euclidean space. Suppose the coordinate system of the Galilean space is considered to be the Euclidean system [16]. In that case, some of our results on isometry are a generalization of the concept of "isometry of surfaces along a section" studied in the works of A. Sharipov [17, 18].

The notion of isometry in the metric of Galilean space also differs from this notion in Euclidean space. The main reason is that the distance is defined differently.

In this article, we obtain a development of a surface that uniquely projects onto a plane in general position, study isometric and completely isometric surfaces, and prove conditions for isometric and completely isometric surfaces of the Galilean space.

2 Preliminaries

Let a three-dimensional affine space A_3 be given, $Oxyz$ is a system of affine coordinates with origin at the point $O(0,0,0)$ and $\{\vec{i},\vec{j},\vec{k}\}$ are basis vectors in this space.

The scalar product of vectors $\vec{X}\{x_1,y_1,z_1\}$ and $\vec{Y}\{x_2,y_2,z_2\}$ is determined by the formula,

$$(\vec{X}\vec{Y}) = \begin{cases} x_1x_2, & if\ x_1x_2 \neq 0, \\ y_1y_2 + z_1z_2, & if\ x_1x_2 = 0. \end{cases} \tag{1}$$

Definition 1.1 An affine space in which the scalar product of vectors $\vec{X}$, $\vec{Y}$ is defined by formula (1) is called a Galilean space and is denoted by G_3.

The scalar product (1) is called the degenerate scalar product. The degenerate scalar product of vectors appears in pseudo-Euclidean spaces [3] due to the isotropy of vectors.

Let points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ be points of the Galilean space G_3, and $x_1 \neq x_2$. Then the vector $\overrightarrow{AB}$

$$\overrightarrow{AB}\{x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1\}$$

and

$$AB = \left|\overrightarrow{AB}\right| = \sqrt{(\overrightarrow{AB}\cdot\overrightarrow{AB})} = |x_2 - x_1|\,.$$

The distance between points A and B is equal to the length of the projection of the vector $\overrightarrow{AB}$ onto the axis Ox (see Fig. 1).

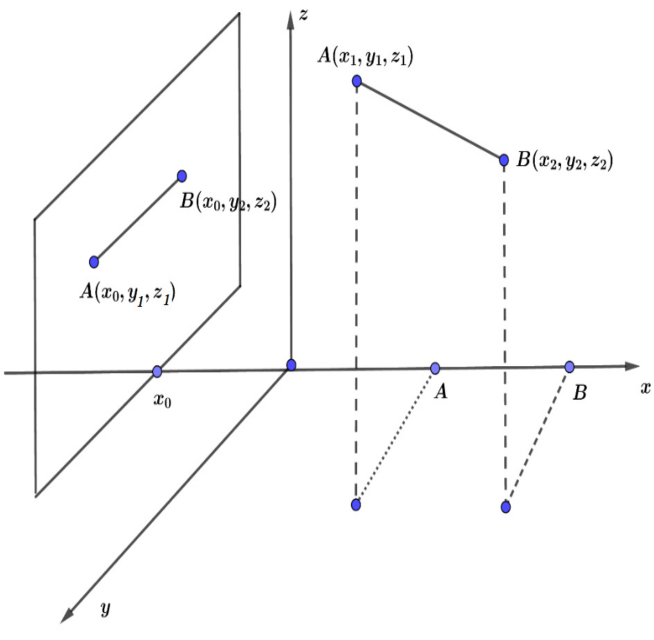

Figure 1. Distance between points A and B.

If $x_1 = x_2 = x_0$, then vector $\overrightarrow{AB}$ is parallel to plane Oyz, and the distance between points $A(x_0,y_1,z_1)$ and $B(x_0,y_2,z_2)$ is determined by the formula

$$AB = |\overrightarrow{AB}| = \sqrt{(y_2-y_1)^2 + (z_2-z_1)^2}.$$

Obviously, points A and B lie on the plane $x = x_0$, and the distance will be the Euclidean distance between the corresponding points. Therefore, the geometry on the plane $x = x_0$ of the Galilean space will be Euclidean, such planes are called special planes of the Galilean space [3]. We consider curves that do not have more than one point with special planes, and surfaces that do not have special tangent planes.

The motion of the Galilean plane is a linear transformation:

$$\begin{cases} x' = x + a \\ y' = hx + y + b \end{cases} \qquad -\infty < h < +\infty$$

consisting of a parallel transfer to a vector $\vec{a} = (a;b)$ and a transformation matrix $A = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$, where $DetA = 1$ [19, 20]. Matrix A will be an element of the Heisenberg group [21, 22]. When in linear transformation $a = b = 0$, then

$$\begin{cases} x' = x \\ y' = hx + y. \end{cases}$$

If $x = x_0$ is a straight line parallel to the axis Oy, then the linear transformation will have the following form:

$$\begin{cases} x' = x_0 \\ y' = hx_0 + y. \end{cases}$$

$x' = x_0$ means that the straight line does not change, and from equality $y' = hx_0 + y$ the straight-line slides at a distance hx_0 along the straight line itself (see Fig. 2).

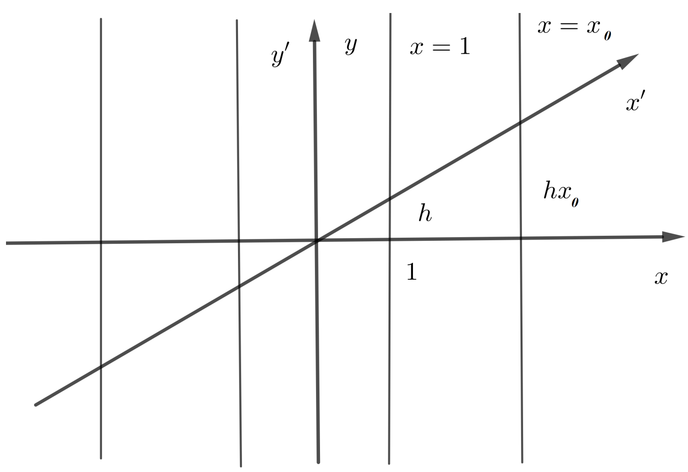

Figure 2. Linear Transformation.

Let F be a surface of space G_3 without special tangent planes. We introduce a special system of curvilinear coordinates. To do this, consider all possible intersections F with special planes $x = const$.

We choose as curvilinear coordinates $u = u_0$, a family of curves formed by intersections of the surface with special planes, and as coordinate lines $v = v_0$ we choose arbitrary lines that form a network on the surface F. With this choice of curvilinear coordinates, the surface equations will have the form [3],

$$\vec{r} = \vec{r}(u,v) = u\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}. \quad (2)$$

In this case, vectors $\vec{r}_u$, $\vec{r}_v$ form a basis in the tangent plane of the surface, which is Galilean. The direction of the vector $\vec{r}_v$ corresponds to the selected direction of the Galilean plane.

The vector equation of a line in Galilean space will look like as:

$$\vec{r} = \vec{r}(s) = s\vec{i} + y(s)\vec{j} + z(s)\vec{k}.$$

Let a curve with equation $v = v(u)$ be given on the surface F. Consider the length of the curve on the surface. Calculating the arc length of a curve segment with ends at points $A(u_0)$ and $B(u_1)$, where $u_1 \neq u_0$, we obtain that the differential of the arc length is $ds = |\vec{r}_u du + \vec{r}_v dv|$. Therefore, the square of the differential of the curve arc on the surface is equal to the square of the increment of the coordinate,

$$ds^2 = du^2.$$

The resulting form is called the first fundamental form of the surface. When $du = 0$, we have $u = const$. In this case, the curve lies on a special plane. The curve arc length differential is calculated by the formula,

$$ds_2^2 = (y_v^2 + z_v^2)dv^2 = G(u,v)dv^2$$

where ds_2^2 is the first additional fundamental form of the surface. Therefore, for the chosen curvilinear coordinate, the coefficients of the first fundamental form have the form $E = 1$, $G = y_v^2 + z_v^2$.

Let two generic curves emerge from the point $M(u_0, v_0)$ to the surface, i.e., curves do not have a tangent parallel to special planes. The corresponding differentials of the radius vectors will be denoted by $d\vec{r}$ and $\delta\vec{r}$. The angle θ between the curves is defined as the angle between vectors $d\vec{r}$ and $\delta\vec{r}$ therefore,

$$\theta = \sqrt{G(u,v)}\left(\frac{dv}{du} - \frac{\delta v}{\delta u}\right).$$

Similarly, to the Euclidean case, one can introduce the concept of surface domain. Let F be a smooth surface and D be a domain on it. The surface domain is determined by the formula,

$$S = \iint_D \sqrt{G(u,v)}dudv.$$

Let F be a regular surface given by the vector function (2) in the Galilean space G_3.

The normal of the tangent plane is the vector of the special plane orthogonal to the vector $\vec{r}_v$. Then the unit normal vector is determined by formula

$$\vec{n} = \pm\frac{z_v\vec{e_2} - y_v\vec{e_3}}{\sqrt{{y_v}^2 + {z_v}^2}} = \pm\frac{z_v\vec{e_2} - y_v\vec{e_3}}{\sqrt{G(u,v)}}.$$

The choice of sign corresponds to fixing the internal or external normal.

The second fundamental form of the surface is called,

$$II = (d^2\vec{r}\vec{n}) = Ldu^2 + 2Mdudv + Ndv^2,$$

where

$$L = (\vec{r}_{uu}\vec{n}) = \frac{y_{uu}z_v - z_{uu}y_v}{\sqrt{G(u,v)}},$$
$$M = (\vec{r}_{uv}\vec{n}) = \frac{y_{uv}z_v - z_{uv}y_v}{\sqrt{G(u,v)}},$$
$$N = (\vec{r}_{vv}\vec{n}) = \frac{y_{vv}z_v - z_{vv}y_v}{\sqrt{G(u,v)}}.$$

Half the value of the second fundamental form of the surface expresses the main part of the deviation of the surface point from the tangent plane. The distance from a surface point to a tangent plane is measured along a special plane.

In [3], derivational surface formulas are defined, which are analogous to the Frenet formulas. At each point of the regular surface given by formula (2), there are three linearly independent vectors $\vec{r}_u$, $\vec{r}_v$, $\vec{n}$, moreover, vector $\vec{r}_u$ is spatial, and $\vec{r}_v$ and $\vec{n}$ are parallel to the special plane. Vectors $\vec{r}_{uu}$, $\vec{r}_{uv}$, $\vec{r}_{vv}$, as well as $\vec{n}_u$, $\vec{n}_v$ can be expanded in terms of basis vectors $\vec{r}_u$, $\vec{r}_v$, $\vec{n}$.Considering the parallelism of these vectors of the special plane, analogs of the derivational formulas are defined by:

$$\begin{aligned}
\vec{r}_{uu} &= \Gamma_{11}^2\vec{r}_v + L\vec{n},\\
\vec{r}_{uv} &= \Gamma_{21}^2\vec{r}_v + M\vec{n},\\
\vec{r}_{vv} &= \Gamma_{22}^2\vec{r}_v + N\vec{n},\\
\vec{n}_u &= -\tfrac{M}{G}\vec{r}_v,\\
\vec{n}_v &= -\tfrac{N}{G}\vec{r}_v.
\end{aligned}$$

Values Γ_{ij}^2 – analogues of the Christoffel coefficients – have the form:

$$\Gamma_{11}^2 = \frac{F_u - \frac{1}{2}E_v}{G}, \quad \Gamma_{12}^2 = \frac{G_u}{2G}, \quad \Gamma_{22}^2 = \frac{G_v}{2G}, \quad (3)$$

where

$$F = y_u y_v + z_u z_v \ , \qquad E = y_u^2 + z_u^2.$$

The condition for the integrability of derivational formulas are the analogs of the Peterson-Codazzi equations:

$$\begin{cases} L_v - M_u = \Gamma_{12}^2 M - \Gamma_{11}^2 N, \\ N_u - M_v = \Gamma_{12}^2 N - \Gamma_{22}^2 M, \end{cases}$$

and Gauss

$$K = \frac{LN - M^2}{G} = \frac{1}{\sqrt{G}}\left(\frac{F_u - \frac{1}{2}E_v}{\sqrt{G}}\right)_v - \frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{du^2}. \quad (4)$$

The expression $\frac{F_u - \frac{1}{2}E_v}{\sqrt{G}}$ is called defect curvature of the surface,

$$D(u,v) = F_u - \frac{1}{2}E_v.$$

Then $D(u,v) = \vec{r}_{uu}\cdot\vec{r}_v = |\vec{r}_{uu}|\cdot|\vec{r}_v|\cos\varphi$, $|\vec{r}_v| = \sqrt{G(u,v)}$, where φ is the angle between vectors $\vec{r}_{uu}$ and $\vec{r}_v$ on the special plane [3, 23].

3 Main Results

3.1 Development a surface onto a plane

By analogy with Euclidean space [3], we define the concept of a surface development in the Galilean space. The properties of the Galilean space metric make it possible to develop a surface onto a plane in a way that the distance between two points on the surface and the corresponding points on the plane would be of the same order and would be equal in magnitude. In this case, the location of the surface relative to the special plane plays an important role, because when the surface is developed by a section along the special plane, points on the special straight plane correspond. For convenience, we take plane Oxy in space G_3 as the development plane.

Definition 3.1.1. If there is a unique mapping between the points of the surface $F \subset G_3$ and the points of the domain G on the plane Oxy, the distances between the corresponding points are of the same order and are equal, then the domain G is called the development of the surface F on the plane Oxy.

In Euclidean space, only convex polyhedra, cylindrical surfaces, and cones have a development. The degeneracy of the Galilean space metric makes it possible to develop surfaces of a wider class.

Theorem 3.1.1. The surface $F \in G_3$ of width $[a, b]$ and uniquely projected onto the plane Oxy has a development G on the strip $a \leq x \leq b$ of the plane Oxy.

Proof. First, consider the surface $F \subset G_3$ with the boundary L and a single-valued projection onto the domain D of the plane Oxy, and the points of the boundary L are projected onto the boundary ∂D. Let us assume that A and B are the points of the surface F, in which the planes $x = a$ and $x = b$ are the reference planes of the surface. Then, by points A and B, edge L splits into two curves L_1 and L_2.

The projections of points A and B in the plane Oxy will be denoted by A^* and B^*. Then the boundary ∂D of the domain D is divided into two parts by points A^* and B^*.

Let ∂D_1 be a part of the edge of the domain D, the points of which are the projection of the curve L_1, that is, the points of the edge L_1 of the surface F are uniquely projected onto the points ∂D_1.

Let us consider the rays on the plane Oxy, whose origins are the points ∂D_1 and directed towards the domain D. On these rays we set aside segments with a length equal to the length of the section of the surface F with a special plane corresponding to the direction of the ray. Then on the strip $a \leq x \leq b$ a certain domain G is formed with a width $[a, b]$ and for each plane $x = x_0$ with a length equal to the length of the section of the surface F by this special plane.

In contrast to Euclidean space, any surface that uniquely projects onto a generic plane can be developed. This unfolding is isometric to the surface in the sense of Galilean space.

Let D be a domain on the plane in general position Oxy, where$D = \{(x, y) \in G_2 : a \leq x \leq b; \varphi_1(x) \leq y \leq \varphi_2(x)\}$, where $\varphi_1(x), \varphi_2(x)$ are continuous functions in $[a, b]$.

Let us consider a surface $F : z = f(x, y) \quad (x, y) \in D$ with a boundary projected uniquely onto the boundary of domain D.

Theorem 3.1.2. The surface $F : z = f(x, y)$ is development onto the domain $G = \{(x, y) \in G_2 : a \leq x \leq b; 0 \leq y \leq \int_{\varphi_1}^{\varphi_2} \sqrt{1 + f_y^2(x, y)}dy\}$ on the plane Oxy.

Proof. Obviously, the width of the surface F is equal to the width of the domain D, that is, $a \leq x \leq b$ is the width of F. In isometry, surfaces with equal widths are considered.

When $x = x_0 \in [a, b]$, the section of the surface F by the special plane $x = x_0$ is some curve $l(x_0)$ on this special plane. The length of this curve is calculated by formula,

$$\varphi(x_0) = \int_{\varphi_1(x_0)}^{\varphi_2(x_0)} \sqrt{1 + f_y^2(x_0, y)}dy.$$

Considering as curve $l(x_0)$ a segment with origin at point $(x_0, 0)$ and length equal to $\varphi(x_0)$, where x_0 is a point from $[a, b]$, we obtain an isometric domain G on plane Oxy, for which the following holds true condition:

$$G = \{(x, y) \in R_2^1 : a \leq x \leq b;$$

$$0 \leq y \leq \int_{\varphi_1(x_0)}^{\varphi_2(x_0)} \sqrt{1 + f_y^2(x_0, y)}dy\}.$$

The theorem has been proven.

Let us give an example of a development of the surface of the Galilean space that satisfies the condition of the theorem.

Example 3.1.1. Let $D : x^2 + y^2 = 1$ and function $z = \sqrt{1 - x^2 - y^2}$. Then domain G is a semi-ellipse on the plane Oxy given by equation $y = \pi\sqrt{1 - x^2}(seeFig.3)$.

Indeed, the width of the domain $-1 \leq x \leq 1$.

We calculate the integral

$$\varphi(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + \frac{y^2}{1 - x^2 - y^2}}dx = \pi\sqrt{1 - x^2}.$$

Consequently, the surface $z = \sqrt{1 - x^2 - y^2}$ is isometric on the flat domain bounded by the abscissa axis to the semi-ellipse given by equation $y = \pi\sqrt{1 - x^2}$.

It is known [24] that a sphere in Euclidean space is inflexible.

3.2 Isometry of surfaces in G_3

Let F be a bounded surface of the Galilean space. From the general boundedness of the surface follows the boundedness along the axis Ox. Therefore, the special planes given by equation $x = x_i$ intersecting the surface F are also limited, that is, there are numbers a and b such that $a \leq x_i \leq b$. Moreover, the plane $x = a$ limits the surface F on the left along the axis Ox, and the plane $x = b$ - on the right. Numbers a and b can be unlimited.

Definition 3.2.1. The interval $[a, b]$ is called the width of the surface F in the Galilean space.

Definition 3.2.2. Semi-isometric surfaces are surfaces that have equal widths.

There is a reasonably broad class of semi-isometric surfaces. In addition, a one-to-one correspondence can always be established between semi-isometric surfaces such that the distances between the corresponding special planes intersecting the surface F will be equal.

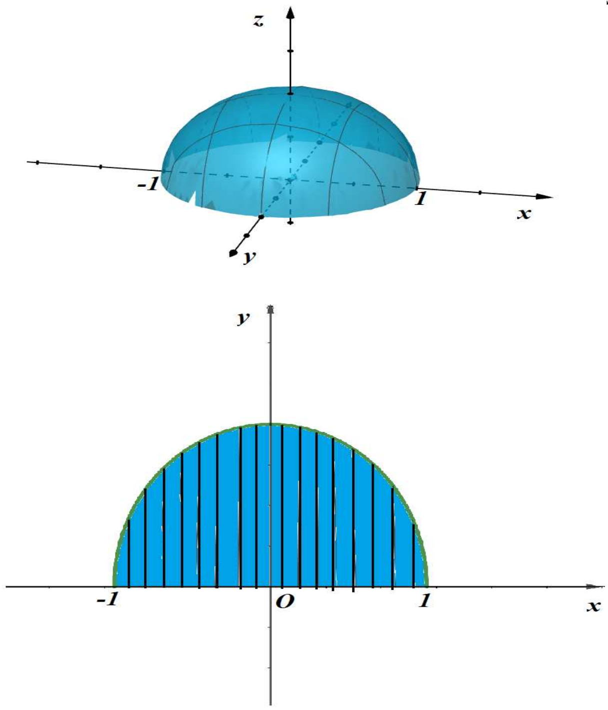

Figure 3. Development of Surface.

Definition 3.2.3. Semi-isometric surfaces are called isometric if, in the corresponding sections of special planes, the mappings are isometric.

When the surfaces are semi-isometric, that is, equal to their widths, then in equation (2) the parameters can be chosen to be the same for both surfaces. The isometry of the section means that, under $u = u_0$, equality,

$$\int_{v_0}^{v} \sqrt{G_1(u_0, v)}dv = \int_{v_0}^{v} \sqrt{G_2(u_0, v)}dv.$$

is satisfied. It follows from this equality that $G_1(u_0, v) = G_2(u_0, v)$, that is, the coefficients of the first fundamental forms of the surfaces are equal.

Therefore, we can conclude that it is possible to choose the coordinate lines of isometric surfaces so that they have the same first fundamental forms.

Let us give an example of isometric surfaces. Consider the surfaces given by equations,

$$F_1: \quad z = \frac{1}{2}(x^2 + y^2); F_2: \quad z = \frac{1}{2}(2x + y^2).$$

Let's calculate the coefficients of the first fundamental forms, as well as the curvature defect of surfaces F_1 and F_2. The coefficients of the first fundamental forms of surfaces $G_1(x, y) = G_2(x, y) = 1+y^2$ are equal. The curvature defects of surfaces $D_1(x, y) = x$ and $D_2(x, y) = 0$ are not equal. Therefore, there are surfaces with various curvature defects.

For a geometric representation, consider an example of semi-isometric but not isometric surfaces in Galilean space.

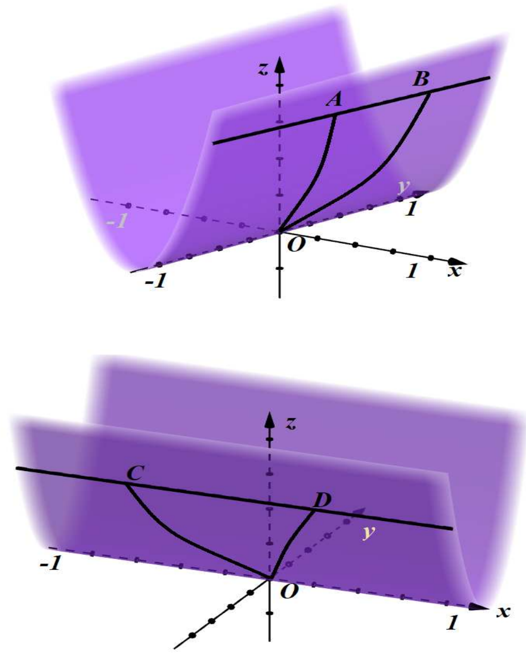

Figure 4. Surfaces.

Consider two cylindrical surfaces $z = x^2$ and $z = y^2$ defined in domain $D\{-1 \leq x \leq 1, \ -1 \leq y \leq 1\}$.

If we consider these surfaces as surfaces of Euclidean space, then it is easy to see that they are equal. Transformation $x' = y, \ y' = x$ can transform the first surface into the second one. Obviously, this equality implies that they are isometric.

In Galilean space, these surfaces are semi-isometric, but not isometric. Of course, these surfaces are not equal.

We will show the proof of these arguments in the figures (see Fig. 4 and 5).

First, consider the vector equations of these surfaces and their graph: Surfaces are given by equations,

$$\vec{r_1}(u, v) = u\vec{i} + v\vec{j} + u^2\vec{k} \ ; \qquad \vec{r_2}(u, v) = u\vec{i} + v\vec{j} + v^2\vec{k}\,.$$

Their domain of definition is the domain $D\{-1 \leq u \leq 1, \ -1 \leq v \leq 1\}$.

Consider the bending of these surfaces onto the plane, that is, a unique mapping of the surface onto the plane that preserves the distance between the corresponding points and the order.

The first surface $\vec{r_1}(u, v)$ is mapped onto the D, domain which is the domain of the surface function. The second surface $\vec{r_2}(u, v)$ is mapped onto the domain $D^*\{-1 \leq u \leq 1, \ -l \leq v \leq l\}$. Here $l = \int_{-1}^{1} \sqrt{1 + 4v^2}dv$ is the length of the parabola.

A feature of the isometry of the surface of the Galilean space can be seen in the mapping of triangles OAB, OCD onto the corresponding surfaces. In Galilean space, triangles OAB and OA^*B^*, as well as OCD and OC^*D^* are equal to each other.

For comparison, recall that in Euclidean space a metric with positive Gaussian curvature uniquely defines a convex surface.

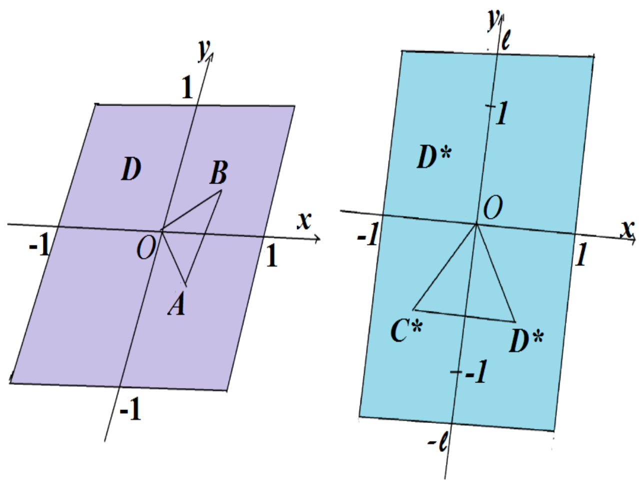

Figure 5. Development of Surface.

Therefore, the isometry of surfaces in the Galilean space is not sufficient for solving problems about the unique definiteness of a surface. In this connection, we introduce the notion of complete isometry.

Definition 3.2.4. Isometric surfaces are called completely isometric if their curvature defects are equal at the corresponding points.

Let us be given the Christoffel symbols $\Gamma_{11}^2,\ \Gamma_{12}^2,\ \Gamma_{22}^2$ of the surface of the Galilean space.

Lemma 3.2.1. If differentiable functions $\mathrm{X}(u,v),\ \mathrm{H}(u,v)$ are given, which are the Christoffel coefficients of the surface F in G_3, then the coefficient of the first fundamental form of the surface F can be found.

Proof. Let $\Gamma_{12}^2=\mathrm{X}(u,v)$ and $\Gamma_{22}^2=\mathrm{H}(u,v)$ be given differentiable functions that are, respectively, the Christoffel coefficients of some surface F from G_3. Then by the formula of the Christoffel coefficients

$$\begin{cases} \frac{G_u(u,v)}{2G(u,v)}=\mathrm{X}(u,v), \\ \frac{G_v(u,v)}{2G(u,v)}=\mathrm{H}(u,v). \end{cases}$$

It is easy to check that,

$$\frac{\partial \mathrm{X}(u,v)}{\partial v}=\frac{\partial \mathrm{H}(u,v)}{\partial u}=\frac{G_{uv}G-G_uG_v}{2G^2}.$$

It follows that the differential equation

$$\mathrm{X}(u,v)du+\mathrm{H}(u,v)dv=0,$$

is a differential equation in total differentials. Solving this equation by the known method [25], we obtain function $G(u,v)$ – the coefficient of the first fundamental form of the surface F in G_3.

Theorem 3.2.1. Surfaces with equal coefficients $\Gamma_{12}^2,\ \Gamma_{22}^2$ are isometric.

The proof of the theorem follows from Lemma 3.2.1. When $\Gamma_{12}^2,\ \Gamma_{22}^2$ is given, coefficient $G(u,v)$ is uniquely determined. When the coefficients of the first fundamental form $G(u,v)$ are equal for two surfaces, these surfaces are isometric.

Theorem 3.2.2. Surfaces with equal coefficients $\Gamma_{11}^2,\ \Gamma_{12}^2,\ \Gamma_{22}^2$ are completely isometric.

Proof. When the coefficients $\Gamma_{12}^2,\ \Gamma_{22}^2$ are equal, the coefficient of the first fundamental form $G(u,v)$ can be determined. Using the coefficient Γ_{11}^2, one can determine $D(u,v)$ - the surface curvature defect. Under given initial conditions, these coefficients are uniquely determined. This means that the Christoffel symbols are equal, that is, for their equality it is sufficient that the coefficients of the first fundamental forms and surface defects are equal. As a definition, surfaces are completely isometric.

Lemma 3.2.2. The Gaussian curvature of the surface of a Galilean space is completely determined by one of the Christoffel symbols Γ_{12}^2:

$$K=-\left(\Gamma_{12}^2\right)^2-\left(\Gamma_{12}^2\right)_u, \quad if\ \ D(u,v)=0. \tag{5}$$

Proof. From the following equality $\Gamma_{12}^2=\frac{G_u}{2G}$ we get the derivative according to u:

$$\left(\Gamma_{12}^2\right)_u=\left(\frac{G_u}{2G}\right)_u=\frac{G_{uu}G-G_u^2}{2G^2}=$$

$$=-\frac{G_u^2-2G_{uu}G}{4G^2}-\frac{G_u^2}{4G^2}=-K-\left(\Gamma_{12}^2\right)^2$$

if we find the Gaussian curvature K from the last equation, we get the equation (5):

$$K=-\left(\Gamma_{12}^2\right)^2-\left(\Gamma_{12}^2\right)_u.$$

If $D(u,v)=0$ is the case, it is reminiscent of the metric in the semi-geodesic coordinate system on the surface in the Euclidean space.

Lemma 3.2.3. The Gaussian curvature of the surface of the Galilean space, if $D(u,v)\neq 0$, then is completely determined by the Christoffel symbols:

$$K=\left(\Gamma_{11}^2\right)_v+\Gamma_{22}^2\Gamma_{11}^2-\left(\Gamma_{12}^2\right)_u-\left(\Gamma_{12}^2\right)^2. \tag{6}$$

Proof. Let us be given Christoffel symbols (3) with equality. Let's write the Gaussian curvature from this equation (4) as follows:

$K=\frac{1}{\sqrt{G}}\left(\frac{D(u,v)}{\sqrt{G}}\right)_v-\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{du^2}=\frac{2D_v(u,v)G-G_vD(u,v)}{2G^2}-\frac{G_u^2-2G_{uu}G}{4G^2}=\frac{D_v(u,v)}{G}-\frac{G_v}{2G}\cdot\frac{D(u,v)}{G}-\frac{G_u^2-2G_{uu}G}{4G^2}$ that is

$$K=\frac{D_v(u,v)}{G}-\frac{G_v}{2G}\cdot\frac{D(u,v)}{G}-\left(\Gamma_{12}^2\right)^2-\left(\Gamma_{12}^2\right)_u \tag{7}$$

Here $\varphi=D(u,v)=F_u-\frac{1}{2}E_v$ is the defect curvature.

From the following equality $\Gamma_{11}^2=\frac{D(u,v)}{G}$ we get the derivative according to v:

$$\left(\Gamma_{11}^2\right)_v=\left(\frac{D(u,v)}{G}\right)_v=\frac{D_v(u,v)G-G_vD(u,v)}{G^2}=\frac{D_v(u,v)}{G}-$$
$$-\frac{G_v}{G}\cdot\frac{D(u,v)}{G}=\frac{D_v(u,v)}{G}-\frac{G_v}{2G}\cdot\frac{D(u,v)}{G}-\frac{G_v}{2G}\cdot\frac{D(u,v)}{G}.$$

Let's write the last equation as,

$$\frac{D_v(u,v)}{G}-\frac{G_v}{2G}\cdot\frac{D(u,v)}{G}=\left(\Gamma_{11}^2\right)_v+\frac{G_v}{2G}\cdot\frac{D(u,v)}{G}=$$

$$= \left(\Gamma_{11}^2\right)_v + \Gamma_{22}^2\Gamma_{11}^2.$$

Then if we use (7), we get (6),

$$K = \left(\Gamma_{11}^2\right)_v + \Gamma_{22}^2\Gamma_{11}^2 - \left(\Gamma_{12}^2\right)_u - \left(\Gamma_{12}^2\right)^2.$$

So, if Christoffel symbols are given in Galilean space, the Gaussian curvature of the surface can be found.

4 Conclusions

The article is theoretical in nature. The results and methods presented in the article can be used in the preparation of special courses on differential geometry and topology, and are also used in scientific research of staff and students.

This article is devoted to the development and isometry of surfaces in Galilean space G_3. The main results are as follows:

*The concepts of semi-isometry, isometry, and complete isometry in Galilean space are introduced.

*The concept of surface unfolding is generalized for Galilean space.

*Isometric and completely isometric surfaces are studied.

*Conditions for the isometricity of surfaces in the Galilean space are determined.

*A development of the surface is obtained, which is uniquely projected onto a plane in a general position.

*The conditions for isometricity and complete isometricity of surfaces of the Galilean space are proved.

Acknowledgements

We want to express our gratitude to Prof. Dr. Abdullaaziz Artykbaev for all the guidance, support, and instruction he provided us throughout our studies.

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