The Relation between the Spaces $L^1(F)$ and $L^1(\mathbb{T})$ with Some Applications

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Abstract: In this research we find the relation between the nonstandard space of Lebesgue integrable functions $L^1(F)$, where $F = \{\left\lfloor -\frac{N}{2} \right\rfloor + 1, \left\lfloor -\frac{N}{2} \right\rfloor + 2, \dots, 0, \dots, \left\lfloor \frac{N}{2} \right\rfloor\}$ is a *finite set for $N > \mathbb{N}$ and the space of Lebesgue integrable functions $L^1(\mathbb{T})$, where $\mathbb{T} = [-\pi, \pi]$, with some applications by using methods and techniques of nonstandard analysis.

1. Introduction

Let $F = \{\left\lfloor -\frac{N}{2} \right\rfloor + 1, \left\lfloor -\frac{N}{2} \right\rfloor + 2, \dots, 0, \dots, \left\lfloor \frac{N}{2} \right\rfloor\}$. Then F is a nonstandard *finite set for unlimited nonstandard natural number $N > \mathbb{N}$, where $\mathbb{N}$ is the set of standard natural numbers. If N is even, then $F = \{-\frac{N}{2} + 1, -\frac{N}{2} + 2, \dots, 0, \dots, \frac{N}{2}\}$ and if N is odd, then $F = \{-\frac{N-1}{2}, -\frac{N-1}{2} + 1, \dots, 0, \dots, \frac{N-1}{2}\}$. Without loss of generality, we assume that N is even through this work.

The set F with the addition operation modulo N is a cyclic group as an algebraic structure. The set $\mathbb{T} = [-\pi, \pi]$ is the additive circle group modulo 2π. The *finite set F is an internal nonstandard model of the closed interval $\mathbb{T}$ in the standard real numbers $\mathbb{R}$. So, $L^1(F)$ is the nonstandard space of Lebesgue integrable functions on F. Also, an internal function f on F is Lebesgue integrable if f is both S-integrable and almost S-continuous on F as defined by Cartier and Perrin (1995). The elements of the space $L^1(\mathbb{T})$ are equivalence classes of Lebesgue integrable functions on $\mathbb{T}$ such that f and g are in the same class if $f = g$ almost everywhere on $\mathbb{T}$, (see Katznelson, 2004).

The interest object of study in this paper is modeling every function in the nonstandard universe $L^1(F)$ by a function in $L^1(\mathbb{T})$ by using nonstandard means (Robinson, 1996). This paper is, initially, about the converse of the theorem proved by Lak (2015), which says that "For each $g \in L^1(\mathbb{T})$ there is $G \in L^1(F)$ such that $g(\mathrm{st}\left(\frac{2\pi w}{N}\right)) = {}^\circ G(w)$ for almost all $w \in F$ " as stated and proved in Theorem 2.1. In addition through this work we present theorems related to discrete and continuous Fourier analysis (Walker, 1988) in $L^1(F)$ using nonstandard methods. The discrete Fourier transform (DFT) of complex numbers $f(n)$, $n \in F$ denoted by $\hat{f}(n)$, $n \in F$ is given by $\hat{f}(n) = \frac{1}{N}\sum_{k \in F} f(k) e^{-2\pi i k n/N}$ and the inverse of discrete Fourier transform (IDFT) of complex numbers $\hat{f}(n)$, $n \in F$ are $f(n)$, $n \in F$ defined by $f(n) = \sum_{k \in F} \hat{f}(k) e^{2\pi i k n/N}$ (Cizek, 1986). Towards the end of this paper we present a result concerning the convolution of functions f and g denoted by $f * g$ and defined by $(f * g)(r) = \frac{1}{N}\sum_{s\in F} g(r-s)f(s)$ (Lak, 2015).

2. The Relation Between $L^1(F)$ and $L^1(\mathbb{T})$

2.1 Theorem

If $h \in L^1(F)$, then there is $f \in L^1(\mathbb{T})$ such that $f(t) = \text{st}\left(h\left(\left\lfloor\frac{Nt}{2\pi}\right\rfloor\right)\right)$

almost everywhere on $\mathbb{T}$**.**

Proof. Assume that $h \in L^1(F)$, where F is the *finite set of order $N > \mathbb{N}$. Then h is S-integrable and almost S-continuous internal function on F. So, there is a rare subset E of F such that h is S-continuous on $F - E$.

Let $B = \{ t \in \mathbb{T}: h\left(\left\lfloor\frac{Nt}{2\pi}\right\rfloor\right) \notin {}^*\mathbb{C}_{fin}\}$. Since $\int_F |h| d\mu$ is limited and h is almost S-continuous on F, then B is a measurable subset of $\mathbb{T}$ and it has a Lebesgue measure zero via Loeb Theorem as given in Lindstrom (Cutland, 1988). Then for given $\varepsilon > 0$ in $\mathbb{R}$, there are *finite sets A and C such that $A \subseteq \text{st}^{-1}B \subseteq C$ and $\frac{\text{card}\,(C-A)}{N} < \varepsilon$. In this case

$$\text{st}\left(\frac{\text{card}\,A}{N}\right) = \text{st}\left(\frac{\text{card}\,C}{N}\right) = \lambda(B),$$

which is the Lebesgue measure of B.

Now we define $f : \mathbb{T} \to \mathbb{C}$ as follows

$$f(t) = \begin{cases} \text{st}\left(h\left(\left\lfloor\frac{Nt}{2\pi}\right\rfloor\right)\right) & if\ t \in \mathbb{T} - B, \\ 0 & if\ t \in B. \end{cases}$$

From the definition of f, we deduce that $|f(t)|$ is limited for all $t \in \mathbb{T}$. So, $\int_{\mathbb{T}} |f(t)|\, dt$ is limited. That is, $\| f \|_1 = \int_{\mathbb{T}} |f(t)| dt$ is finite.

Moreover, since h is almost S-continuous on F, then there exists a rare subset E of F such that h is S-continuous on $F - E$. Let $a \in F - E$ (a nonstandard model of $\mathbb{T} - B$).

Then for all $t \in F - E$, if $t \approx a$, then $h(t) \approx h(a)$ is true in $F - E$.

So for $\varepsilon > 0$ arbitrary and standard, without loss of generality, let $\varepsilon = \frac{1}{n}$ for some $n \in \mathbb{N}$. We have to find $\delta > 0$ of the form $1/k$ where $k \in \mathbb{N}$ such that

$$\forall\, t\ (|t-a| < \delta \Rightarrow |h(t) - h(a)| < \tfrac{1}{n})$$

is true in $F - E$. So, for all unlimited $k \in {}^*\mathbb{N}$ we have

$$\forall\, t\ (|t-a| < \tfrac{1}{k} \Rightarrow |h(t) - h(a)| < \tfrac{1}{n})$$

is true in $F - E$. Then,

$$\forall\, t\ (|t-a| < \tfrac{1}{k} \Rightarrow |\mathrm{st}(h(t)) - \mathrm{st}(h(a))| < \tfrac{1}{n})$$

is true in $\mathbb{T} - B$. So,

$$\forall\, t\ (|t-a| < \tfrac{1}{k} \Rightarrow |f(t) - f(a)| < \tfrac{1}{n})$$

is true in $\mathbb{T} - B$. Now, let

$$\theta(k) = (k \in {}^{*}\mathbb{N}) \wedge \left(k = 0 \vee \neg\, \forall t\ \left(|t-a| < \tfrac{1}{k} \Rightarrow |f(t) - f(a)| < \tfrac{1}{n}\right)\right).$$

if we couldn't find $\delta = 1/k$, for $k \in \mathbb{N}$, then θ would define $\mathbb{N}$ in ${}^{*}\mathbb{R}$. Which is contradiction with the fact that $\mathbb{N}$ is not internal in ${}^{*}\mathbb{R}$ (Hurd & Loeb, 1985).

Therefore, f is a continuous function on $\mathbb{T} - B$. Hence, f is continuous almost everywhere on $\mathbb{T}$.

Since $\mathbb{T}$ is a Lebesgue measurable set (Fremlin, 2000), so f is a measurable function on $\mathbb{T}$.

Hence, $f \in L^1(\mathbb{T})$ and $f(t) = \mathrm{st}(h(\lfloor \frac{Nt}{2\pi} \rfloor))$ almost everywhere on $\mathbb{T}$.

3. Some Applications

3.1 Theorem

(Riemann Lebesgue Lemma) If $f \in L^1(F)$ and F is the *finites set of order unlimited $N > \mathbb{N}$, then for every standard real b,

$$^{\circ}\lim_{a\in\mathbb{N}}\left(\frac{2\pi}{N}\sum_{k\in F} f(k)\sin(ak+b)\right) = 0.$$

Proof. If f is a constant function and $f(k) = C$, on F, then

$$\frac{2\pi}{N}\sum_{k\in F} f(k)\sin(ak+b) = \frac{2\pi C}{N}(\sin b + \sin b\,[\cos\ a + \cos 2a + \cdots + \cos\tfrac{N}{2}a\,]) .$$

Multiply both sides of the above equation by $2\sin\frac{a}{2}$

$$\frac{4\pi C}{N}\sin\frac{a}{2}\sum_{k\in F}\sin(ak+b) = \frac{2\pi C}{N}\sin b[2\sin\tfrac{a}{2} + 2\sin\tfrac{a}{2}\ \cos a +$$
$$2\sin\tfrac{a}{2}\cos 2a + \cdots + 2\sin\tfrac{a}{2}\cos(\tfrac{N}{2}a)]$$

So,

$$\frac{4\pi C}{N}\sin\frac{a}{2}\sum_{k\in F}\sin(ak+b) = \frac{2\pi C}{N}\sin b[2\sin\tfrac{a}{2} + \sin\left(a+\tfrac{a}{2}\right) - \sin\tfrac{a}{2} +$$
$$\sin\left(2a+\tfrac{a}{2}\right) - \sin\left(a+\tfrac{a}{2}\right) + \cdots + \sin\left(\tfrac{N}{2}a+\tfrac{a}{2}\right) - \sin\left(\left(\tfrac{N}{2}-1\right)a+\tfrac{a}{2}\right)].$$

Then, $\frac{4\pi C}{N}\sin\frac{a}{2}\ \sum_{k\in F}\sin(ak+b)=\frac{2\pi C}{N}\sin b[2\sin\frac{a}{2}+\sin\left(\frac{N}{2}a+\frac{a}{2}\right)]$.

So, $$\frac{2\pi C}{N}\sum_{k\in F}\sin(ak+b)=\frac{2\pi C}{N}\sin b\ [1+\frac{\sin\left(\frac{N}{2}a+\frac{a}{2}\right)}{2\sin\frac{a}{2}}].$$

Therefore, $$\frac{2\pi}{N}|C\sum_{k\in F}\sin(ak+b)|\le\frac{4\pi}{N}\ |C|\approx 0.$$

Hence, $$^{\circ}\lim_{a\in\mathbb{N}}(\frac{2a}{N}\sum_{k\in F}C\sin(ak+b))=0.$$

Notice that this result is true for every constant function $f(x)=C$ on $\mathbb{T}=[-\pi,\pi]$, so it is also true for every average function $E^{P_n}[f]$ of f on any dissection P_n of F [1]. Now, since $f\in\ L^1(F)$, then for all appreciable number $\varepsilon>0$ there exists a partition P_n of F which is a nice dissection (Cartier & Perrin, 1995) such that the function $E^{P_n}[f]\colon F\to{}^{*}\mathbb{C}$, is the average of f relative to P_n, which is a constant function on each atom A of P_n and $\|\ f-E^{P_n}[f]\ \|_1<\varepsilon/2$, for all $n>N$.

Therefore,

$$\left|\frac{2\pi}{N}\sum_{k\in F}\sin(ak+b)\right|=\left|\frac{2\pi}{N}\sum_{k\in F}\left[f(k)-E^{P_n}[f](k)+E^{P_n}[f](k)\right]\sin(ak+b)\right|$$

$$\le\left|\frac{2\pi}{N}\sum_{k\in F}[f(k)-E^{P_n}[f](k)]\sin(ak+b)\right|+\left|\frac{2\pi}{N}\sum_{k\in F}E^{P_n}[f](k)\sin(ak+b)\right|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

3.2 Lemma

The convergence of the Fourier series of a function f at any point $t\in F$ is determined by the behavior of the nth partial sum

$$S_n(t)=\mathrm{st}\left(\frac{2}{N}\sum_{\mathrm{k}\in\mathrm{F}}\left[{}^{*}f\left(\frac{2\pi}{N}(u+t)\right)+{}^{*}f\left(\frac{2\pi}{N}(u-t)\right)\right]{}^{*}D_n\left(\frac{2\pi}{N}\right)\right)$$

in the limit as $n\in\mathbb{N}$. Moreover, $^{\circ}\lim_{a\in\mathbb{N}}\ S_n(t)$ allows us to study convergence of the series.

Proof. The nth partial sum of the Fourier series of a function f on $\mathbb{T}=[-\pi,-\pi]$ is

$$S_n(t)=\frac{1}{2\pi}\int_{\mathbb{T}}f(x)dx+\sum_{k=1}^{n}\frac{2}{2\pi}\int_{\mathbb{T}}f(x)\cos(kx)\cos(kt)dx+$$

$$\sum_{k=1}^{n}\frac{2}{2\pi}\int_{\mathbb{T}}f(x)\sin(kx)\sin(kt)dx$$

Then,

$$S_n(t)=\frac{1}{2\pi}\int_{\mathbb{T}}f(x)\,[1+\sum_{k=1}^{n}2\cos(kx)\cos(kt)+\sum_{k=1}^{n}2\ \sin(kx)\sin(kt)]\ dx.$$

Now, by using some trigonometric identity the above equation becomes

$$S_n(t)=\frac{1}{\pi}\int_{\mathbb{T}}f(x)\left[\frac{1}{2}+\sum_{k=1}^{n}2\cos\frac{2\pi k(x-t)}{2\pi}\right]dx\ .$$

So, by using the Dirichlet kernel function (Weaver, 1989) in the above equation we obtain

$$S_n(t) = \frac{1}{\pi}\int_{\mathbb{T}} f(x)\, D_n(\frac{2\pi k(x-t)}{2\pi})dx\ .$$

Now for $\mathbb{T} = [-\pi, -\pi]$ and a given standard real $\varepsilon > 0$, there is a standard real $\delta > 0$ such that for all N in the standard world

$$\max_{k\in F} \Delta x_k < \delta \Rightarrow \left|S_n(t) - \frac{1}{\pi}\sum_{k\in F} f(x_k)\, D_n\left(\frac{2\pi(x_k-t)}{2\pi}\right)\Delta x_k\right| < \varepsilon,$$

By taking the partition $P = \left\{x_{-\frac{N}{2}}, x_{-\frac{N}{2}+1}, \dots, x_0, \dots, x_{\frac{N}{2}}\right\}$ of $\mathbb{T} = [-\pi, \pi]$ as a *finite set, such that $-\pi = x_{-\frac{N}{2}} < x_{-\frac{N}{2}+1} < \cdots < x_{\frac{N}{2}} = \pi$ then $\Delta x_k = x_k - x_{k-1} = \frac{2\pi}{N}$, and $x_k = \frac{2\pi k}{N}$ for all $k \in F$.

$$\forall\, N > \frac{2\pi}{\delta} \text{ in } \mathbb{N}\ (\ |S_n(t) - \frac{1}{\pi}\sum_{k\in F} {}^*f\left(\frac{2\pi k}{N}\right) {}^*D_n\left(\frac{2\pi k}{N} - \frac{2\pi t}{N}\right)\left(\frac{2\pi}{N}\right)| < \varepsilon).$$

Now by Transfer Principle (Ponstein, 2002), we have a similar statement in the nonstandard world. Also, $N \in {}^*\mathbb{N} - \mathbb{N}$ is greater than the standard real $2\pi/\delta$. Thus we get

$$\left|S_n(t) - \frac{2}{N}\sum_{k\in F} {}^*f\left(\frac{2\pi k}{N}\right) {}^*D_n\left(\frac{2\pi(k-t)}{N}\right)\right| < \varepsilon.$$

Since $N > \mathbb{N}$ it works for all standard real $\varepsilon > 0$, then

$$S_n(t) \approx \frac{2}{N}\sum_{k\in F} {}^*f\left(\frac{2\pi k}{N}\right) {}^*D_n\left(\frac{2\pi(k-t)}{N}\right).$$

Hence,

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{k\in F} {}^*f\left(\frac{2\pi k}{N}\right) {}^*D_n\left(\frac{2\pi(k-t)}{N}\right)\right).$$

Notice that both functions f and the Dirichlet kernel D_n are periods on the *finite set F. So we shift the above summation and rewrite the latter equation as follows

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{k=-\frac{N}{2}+1}^{0} {}^*f\left(\frac{2\pi k}{N}\right) {}^*D_n\left(\frac{2\pi(k-t)}{N}\right)\right) + \mathrm{st}\left(\frac{2}{N}\sum_{k=1}^{N/2} {}^*f\left(\frac{2\pi k}{N}\right) {}^*D_n\left(\frac{2\pi(k-t)}{N}\right)\right)$$

Now make the substitution of the variables $k - t = u$, then $t = k + u$ which implies that

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{u=-\frac{N}{2}+1}^{0} {}^*f\left(\frac{2\pi}{N}(u+t)\right) {}^*D_n\left(\frac{2\pi}{N}u\right)\right) +$$

$$\mathrm{st}\left(\frac{2}{N}\sum_{u=1}^{N/2} {}^*f\left(\frac{2\pi}{N}(u-t)\right) {}^*D_n\left(\frac{2\pi}{N}u\right)\right).$$

Notice that, in the first sum let $u = -u$, change and invert the limits of the summation. Also, we have the Dirichlet kernel is an even function, that is,

$D_n(u) = D_n(-u)$, for every $u \in F$. Therefore,

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{u=0}^{N/2}\left[{}^*f\left(\frac{2\pi}{N}(u+t)\right) + {}^*f\left(\frac{2\pi}{N}(u-t)\right)\right]{}^*D_n\left(\frac{2\pi}{N}u\right)\right).$$

3.3 Theorem

(Riemann Localization Theorem) The behavior of the Fourier series of the function $h \in L^1(F)$ at any point $t \in F$ depends only on the values of h on $\mathrm{N_r(t)} = \{\mathrm{t-r, t-r+1, t-r+2, \dots, t, \dots, t+r-1, t+r}\}$, for limited $r > 0$.

Proof: Let $h \in L^1(F)$ and define the function $H: F \to {}^*\mathbb{C}$ as follows

$$H(t) = \frac{h\left(\frac{2\pi}{N}(t+k)\right)+h\left(\frac{2\pi}{N}(t-k)\right)}{\sin\frac{\pi k}{N}}.$$

Notice that, $h \in L^1(F)$ and $\frac{1}{2\sin\frac{\pi k}{N}}$ is an S-continuous function and limited for $k > 0$. Thus we have $H \in L^1(\{r, r+1, \dots, \frac{N}{2}\})$.

By writing the nth partial sum of the Fourier series of h with the Dirichlet kernel (Katznelson, 2004) of the form

$$D_n\left(\frac{2\pi k}{N}\right) = \frac{\sin\frac{2\pi\left(n+\frac{1}{2}\right)k}{N}}{\sin\frac{\pi k}{N}},$$

we have

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{k=0}^{N/2}\frac{{}^*h\left(\frac{2\pi}{N}(t+k)\right)+{}^*h\left(\frac{2\pi}{N}(t-k)\right)}{2\sin\frac{\pi k}{N}}\sin\frac{2\pi\left(n+\frac{1}{2}\right)k}{N}\right).$$

or

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{k=0}^{N/2}\mathrm{H(t)}\ \sin\frac{2\pi\left(n+\frac{1}{2}\right)k}{N}\right).$$

Now split the above summation into two summations we obtain

$$S_n(t) = \mathrm{st}\left(\frac{2}{N}\sum_{k=0}^{r}H(t)\sin\frac{2\pi\left(n+\frac{1}{2}\right)k}{N}\right) + \mathrm{st}\left(\frac{2}{N}\sum_{k=r+1}^{N/2}H(t)\sin\frac{2\pi\left(n+\frac{1}{2}\right)k}{N}\right).$$

So, by using Theorem 3.1, we get

$$^\circ\lim_{\mathrm{n}\in\mathbb{N}}\left(\frac{2}{N}\sum_{k=r+1}^{N/2}H(t)\sin\frac{2\pi\left(n+\frac{1}{2}\right)k}{N}\right) = 0.$$

Therefore, from the Lemma 3.2, the result is obtained as

$$^\circ\lim_{\mathrm{n}\in\mathbb{N}} S_n(t) = {}^\circ\lim_{\mathrm{n}\in\mathbb{N}}\left(\frac{2}{N}\sum_{k=0}^{r}\left[h\left(\frac{2\pi}{N}(t+k)\right)+h\left(\frac{2\pi}{N}(t-k)\right)\right]D_n\left(\frac{2\pi k}{N}\right)\right).$$

The discrete Fourier Transform (DFT) (Cizek, 1986) of the product of functions f and g is given by the convolution of the discrete of f and g as shown in the following theorem.

3.4 Theorem

If $f, g \in {}^*\mathbb{C}^F$, then for every $n \in F$, $N\hat{f}(n) * \hat{g}(n) = \widehat{fg}(n)$, where $N = |F|$.

Proof: Notice that $(fg)(n) = f(n)g(n)$

$$= \sum_{k\in F} \hat{f}(k)\, e^{2\pi ikn/N} \sum_{r\in F} \hat{g}(r)\, e^{2\pi irn/N}$$

$$= \sum_{k\in F} \sum_{r\in F} \hat{f}(k)\hat{g}(r) e^{2\pi i(k+r)n/N} .$$

Let $s = k + r$, then $r = s - k$, so we get

$$(fg)(n) = \sum_{k\in F} \sum_{s-k\in F} \hat{f}(k)\hat{g}(s-k)\, e^{2\pi isn/N}.$$

Now, interchange the order of the summation

$$(fg)(n) = N \sum_{s\in F} (\tfrac{1}{N} \sum_{k\in F} \hat{f}(k)\hat{g}(s))\, e^{2\pi isn/N}$$

$$= N \sum_{s\in F} (\hat{f}(k) * \hat{g}(s))\, e^{2\pi isn/N}$$

$$= N \left((\hat{f} * \hat{g})(n) \right)^{-I}.$$

Where $-I$ is the inverse discrete Fourier transform (IDFT). By taking the discrete transform (DFT) (Weaver, 1989) of both sides of the above equation we get the result. That is,

$$N(\hat{f}(n) * \hat{g}(n)) = \widehat{fg}(n).$$

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