Numerical Approximation Method for Solving Differential Equations

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Abstract: This paper investigates numerical methods for solving differential equation. In this work, the continuous least square method (CLSM) was considered to find the best numerical approximation by solving differential equations. The continuous least square method (CLSM) was developed together with the L_2 norm. Numerical results obtained yield minimum approximation error, provide the best approximation. Explicit results obtained are supported by examples treated with MATLAB and Wolfram Mathematica 11.

Keywords: The Least Square Method (LSM), Ordinary Differential Equations, L_2 Norm

1. Introduction

The least square method is a form of mathematical regression analysis used to determine the line of the best fit for a set of data, providing a visual demonstration of the relationship between the data points. Each point of data represents the relationship between a known independent variable and an unknown dependent variable, and the least square method provides the overall rationale for the placement of the line of best fit among the data points being studies. The most common application of this method, which is sometimes referred to as linear or ordinary aims to create a straight line that minimizes the sum of the squares of the errors that are generated by the results of the associated equations, such as the squared residuals resulting from difference in the observed value, and the value anticipated based on that model. And this method of regression analysis begins with a set of a data point to be plotted on an X – and Y- axis graph. An analyst using the least square method will generate a line of best fit that explains the potential relationship between independent and dependent variable and the regression analysis, dependent variable is illustrated on the vertical Y-axis, while independent variables are illustrated on the horizontal X-axis,

The mathematical form of least square method (LSM) for numerical solution of boundary value problems has unlimited application in mathematical physics (Eason, 1976). Based on the application of least square method (LSM), scientists and engineers came up with a solution of complex problems that have many issues like singularity, and difficulties in finding solution (Loghmani, 2008). The method of least square method is a projection method for solving integral and differential equations, in which an approximate solution is found from the condition that the equation is satisfied at some given points from the solution domain (Katayoun, 2014).

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The authors in (Ibrahim, & Rababah, 2020; Rababah, & Ibrahim, 2016a; Rababah, & Ibrahim, 2016b; Rababah, & Ibrahim, 2016c; Rababah, & Ibrahim, 2018) proposed different technique that involve numerical approximation of curves which is an important issue in solving differential equations. Although our main goal is to develop efficient numerical method (CLSM) for solving ordinary differential equations (ODE). This paper considers the continuous least squares approach. Instead of computing integrals or performing discretization, which is usually needed in least squares methods, we establish a least squares objective function based on the control points. The continuous least square method control points can provide best approximation problem.

The project is organized as follows: section 2 briefly explains mathematical preliminaries. Section 3 proposes the continuous least square method for solving differential equation. In section 4, numerical examples for the application of continuous least square method are also provided. And conclusion is given in section 5.

2. Mathematical Preliminaries

Scientists have to tackle many problems using the least square methods (LMS) to solve complex differential equation in finite element method (FEM). In this research, we are going to make use of the following linear boundary value problem, see [1]

$$\mathrm{L(y) = f(x)\ for\ x\ \in\ domain}\ \Omega$$
$$\mathrm{W(y) = g(x)\ for\ x\ \in\ domain}\ \delta\Omega.$$

L is the differential operator, where Ω is the domain in R^1 or R^2 or R^3, and W is the boundary operator. The solution of differential equation by finite element method (FEM) can be expressed in terms of the basic functions of an approximate solution as

$$\tilde{y} = \sum_{i=1}^{n} q_i\, \emptyset^i(X). \qquad [1]$$

Where $\emptyset_i$(X) are weighted basis function, and q_i are coefficients (weights) which can be realized by least square methods. Consider the residual R_L(X) R_W(X) as follows.

$$\mathrm{R_L(x,\tilde{y})\ \ = L\,(\tilde{y}) - f(x)\quad for\quad x\ \in domain}\ \Omega$$
$$\mathrm{R_W(x,\tilde{y}) = w\,(\tilde{y}) - g(x)\quad for\ \ x\ \in boundary}\ \delta\Omega.$$

Substituting y_{exact} solution of the boundary value problem leads to $R_L(\mathrm{x}, y_{exact}) = 0$

and $R_W(\mathrm{x}, y_{exact}) = 0$.

3. Least Square Method for Solving Differential Equation

The aim of this work is to use the continuous least square method (CLSM) and find the coefficients q_i from Eq. (1), this can be achieved by minimizing the error function in L_2 norm which is defined as

$$E = \int_{\Omega} R_L^2\,(x,\tilde{y})dx + \int_{\alpha\Omega} R_W^2(x,\tilde{y})dx. \qquad [2.1]$$

Our goal is to find the best approximate solution that will generate the minimal value of E, and this can be obtained by differentiating Eq. (2.1) with respect to q_i and equating to zero.

$$\frac{\partial E}{\partial q_i} = 0, \qquad for\ i = 1, \dots N,$$

which yields

$$\int_{\Omega} R_L(x, \tilde{y}) \frac{\partial R_L}{\partial q_i} dx + \int_{\alpha\Omega} R_W(x, \tilde{y}) \frac{\partial R_L}{\partial q_i} dx = 0 \quad i = 1, \dots, N. \qquad [2.2]$$

The results obtained by evaluating Eq. (3) that can be expressed in explicit form as

$$Da = b, \qquad [3.3]$$

where D is N x N matrix, $a = [q_1, q_2, q_3, \dots, q_n]^T$, and some column vector b.

4. Application of Least Square Method

In this section, we want to make use of continuous least squares methods (CLSM). The explicit, and algebraic results were obtained from the previous section and applied on the first and second order differential equation.

4.1 Example 1: Using Continuous LSM to Solve First-Order Differential Equation

Consider the first order initial value problem.

$$\left(1 + \sqrt{2}\right)\frac{dy}{dx} + xy = 0, \qquad y(0) = 1, \qquad [4.1]$$

where $0 \leq x \leq 1$. Let

$$L(x, y) = \left(1 + \sqrt{2}\right)\frac{dy}{dx} + xy. \qquad [4.2]$$

Step 1: Choose basis functions. We consider the polynomial.

$$\tilde{y} = \sum_{i=1}^{N} q_i x^i + y_0. \qquad [4.3]$$

Step 2: For $\tilde{y}$ to satisfy the boundary condition, clearly, we must have $y_0 = 1$.

Step 3: From the residual

$$R(x) = \left(1 + \sqrt{2}\right)\frac{d\tilde{y}}{dx} + x\tilde{y}. \qquad [4.4]$$

By replacing $\tilde{y}(x)$ from (4.3) into (4.4), we will get:

$$R(x) = \left(1 + \sqrt{2}\right)\frac{d\left(\sum_{i=1}^{N} q_i x^i + 1\right)}{dx} + x\left(\sum_{i=1}^{N} q_i x^i + 1\right) \qquad [4.5]$$

Step 4: To minimize the square error, we need to set up.

$$E = \int_0^1 R^2(x) dx. \qquad [4.6]$$

The best approximate solution is determined by finding the minimal value of E,

$$\frac{\partial E}{\partial q_i} = 0, \qquad for \quad i = 1, .., N, \tag{4.7}$$

$$\int_0^1 R(x) \frac{\partial R}{\partial q_i} dx = 0, \qquad i = 1, \dots, N, \tag{4.8}$$

or

$$\left(R(x), \frac{\partial R(x)}{\partial q_i} \right) = 0 \quad for \quad i = 1,2,3, \dots \dots, N. \tag{4.9}$$

The solution of Eq. (4.8) or Eq. (4.9) is a linear system which can be used to solve q_i's.

Substituting Eq. (4.5) into Eq. (4.8) for $N = 3$, we obtain the following matrices with the help of Matlab program.

$$D = \begin{pmatrix} 15.2758 & 15.6115 & 15.8053 \\ 15.6115 & 19.6909 & 21.7589 \\ 15.8053 & 21.7589 & 25.3432 \end{pmatrix}, \quad b = \begin{pmatrix} 2.9142 \\ 3.6189 \\ 3.9547 \end{pmatrix}, \quad a = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}. \tag{4.10}$$

And the approximate solution is.

$$\tilde{y} = 0.03795\, x^3 - 0.228417\, x^2 + 0.003400\, x + 1. \tag{4.11}$$

The exact solution is given by.

$$y_{exact} = e^{\frac{x^2}{2} - \frac{x^2}{\sqrt{2}}}. \tag{4.12}$$

The figure below depicts the graph of approximate with exact solutions and error between them for $N = 3$

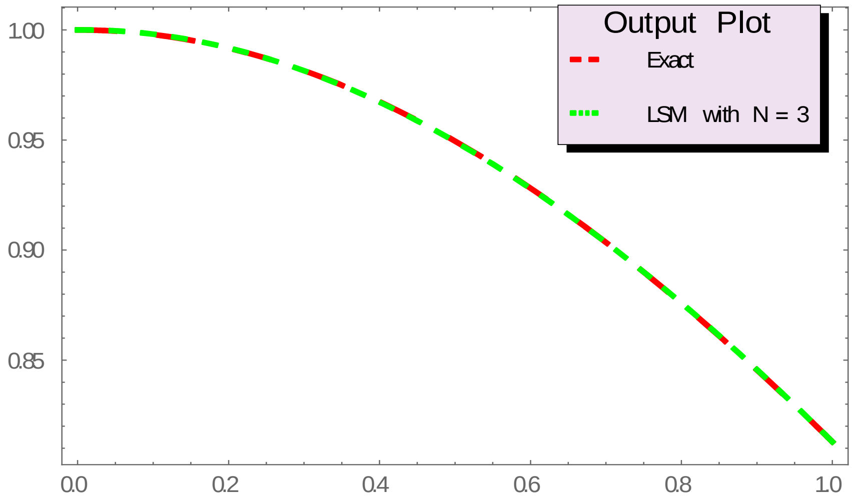

Figure 1: Example of first- order with CLSM for $N = 3$

The error is defined as

$$error = y_{exact} - \tilde{y}. \tag{4.13}$$

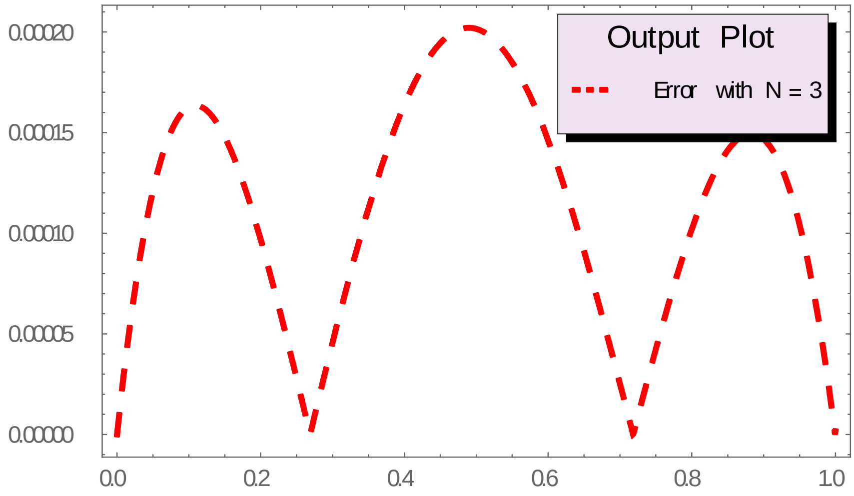

Figure 2: Error of first- order ODE for *N=3*

For $N = 5$, we obtain the following matrices

$$D = \begin{pmatrix} 15.27 & 15.61 & 15.81 & 15.93 & 16.02 \\ 15.61 & 19.69 & 21.76 & 23.01 & 23.85 \\ 15.81 & 21.76 & 25.34 & 27.74 & 29.45 \\ 15.93 & 23.01 & 27.74 & 31.12 & 33.65 \\ 16.02 & 23.85 & 29.45 & 33.65 & 36.92 \end{pmatrix}, \quad b = \begin{pmatrix} 2.91 \\ 3.62 \\ 3.95 \\ 4.15 \\ 4.27 \end{pmatrix}, \quad a = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}. \qquad [4.14]$$

And the approximate solution is.

$$\tilde{y} = -0.0038x^5 + 0.0253x^4 - 0.0018\,x^3 - 0.2067x^2 - 0.00002x + 1. \qquad [4.15]$$

The figure below depicts the graph of approximate with exact solutions and error between them for $N = 5$

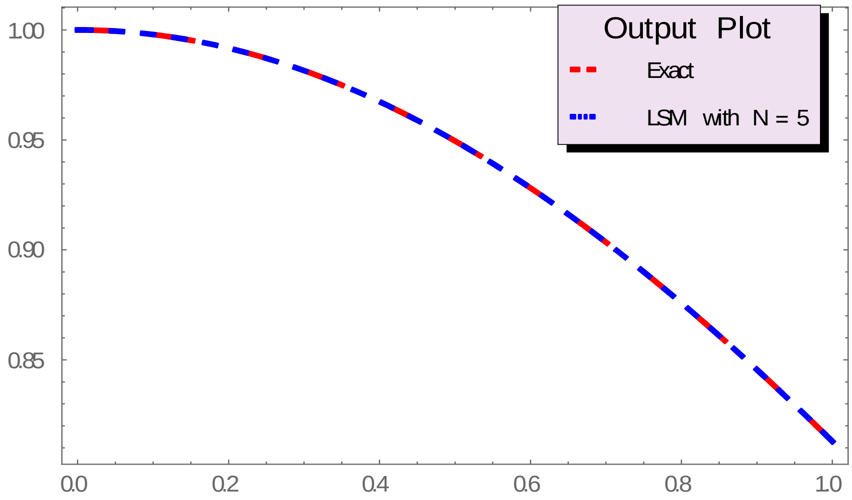

Figure 3: Example of first order with CLSM for *N=5*

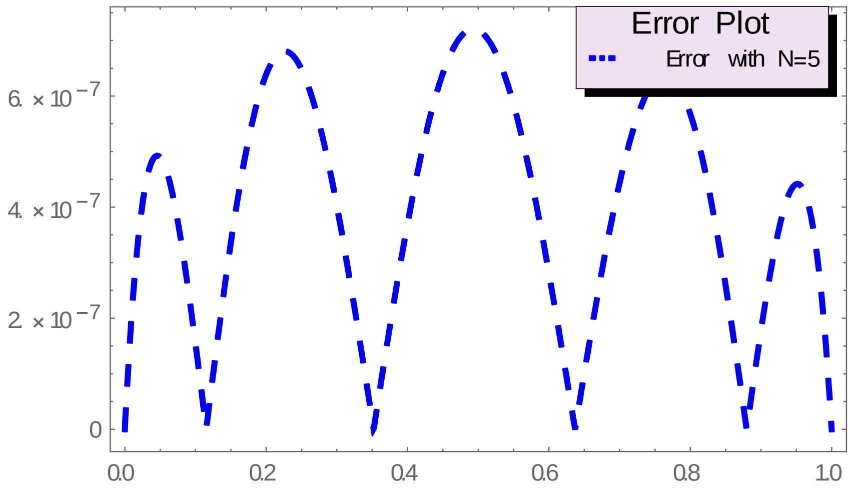

Figure 4: Error of first-order ODE for *N=5*

The figure below depicts the comparison between the graph of exact with approximate solutions, and error for $N = 3$ and that of $N = 5$.

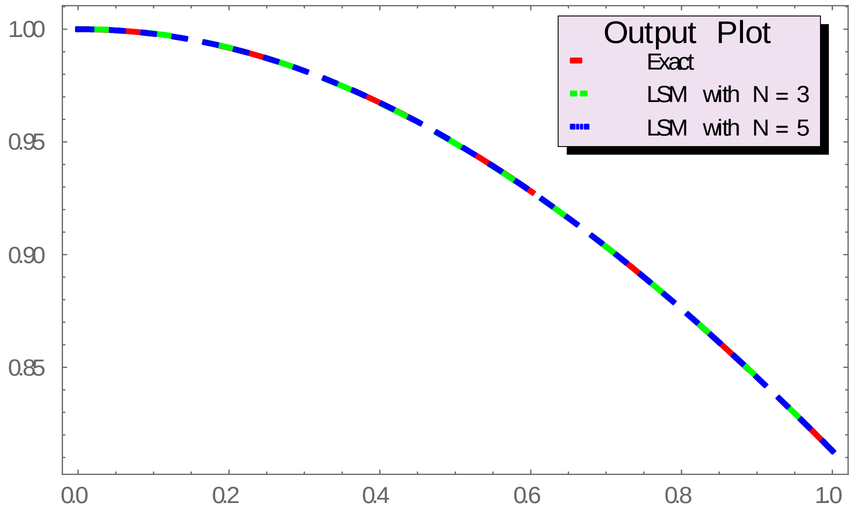

Figure 5: Example of first order with CLSM for *N=3* and that of *N=5*

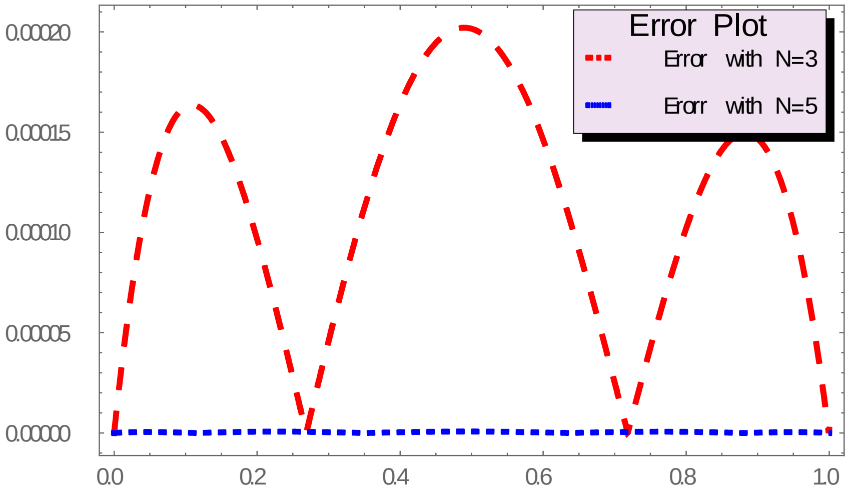

Figure 6: Error of first- order ODE for $N = 3$ and $N=5$

Table 1: Data for the example of first- order ODE with errors for $N=3$ and $N=5$

x	y exact	y approx. for N = 3	y approx. for N = 5	Errors for N = 3	Errors for N = 5
0	1	1	1	0	0
0.1	0.9979	0.9981	0.9979	0.00016	1.5347×10^{-7}
0.2	0.9918	0.9919	0.9918	0.000097	6.3903×10^{-7}
0.3	0.9815	0.9815	0.9815	0.00005	4.0238×10^{-7}
0.4	0.9674	0.9672	0.9674	0.00016	3.7082×10^{-7}
0.5	0.9495	0.9493	0.9495	0.0002	7.1833×10^{-7}
0.6	0.9282	0.9280	0.9282	0.00015	2.8196×10^{-7}
0.7	0.9035	0.9034	0.9035	0.000025	4.4391×10^{-7}
0.8	0.8759	0.8760	0.8759	0.0001	5.6792×10^{-7}
0.9	0.8456	0.8457	0.8456	0.00015	1.7727×10^{-7}
1.0	0.8129	0.8129	0.8129	1.671×10^{-7}	4.0705×10^{-11}

4.2 Example 1: Using Continuous LSM to Solve Second Order Differential Equation

Consider the second-order initial value problem.

$$\frac{d^2y}{dx^2} + \frac{1}{\sqrt{2}}\frac{dy}{dx} + e^x y = 0, \qquad y(0) = 1, \qquad y^I(0) = 0.9 \qquad [4.16]$$

where $0 \leq x \leq 1$. Let

$$L(x, y) = \frac{d^2y}{dx^2} + \frac{1}{\sqrt{2}}\frac{dy}{dx} + e^x y \qquad [4.17]$$

Step 1: Choose basis functions. We consider the polynomial.

$$\tilde{y} = \sum_{i=1}^{N} q_i\, x^i + y_0. \qquad [4.18]$$

Step 2: For $\tilde{y}$ to satisfy the boundary condition, clearly, we must have $y_0 = 1$ and $q_1 = 0.9$.

Step 3: from the residual

$$R(x) = \frac{d^2\tilde{y}}{dx^2} + \frac{1}{\sqrt{2}}\frac{d\tilde{y}}{dx} + e^x\tilde{y}. \qquad [4.19]$$

By replacing $\tilde{y}(x)$ from (4.18) into (4.19), we obtain:

$$R(x) = \frac{d^2\left(\sum_{i=1}^{N} q_i\, x^i + 1\right)}{dx^2} + \frac{1}{\sqrt{2}}\frac{d\left(\sum_{i=1}^{N} q_i\, x^i + 1\right)}{dx} + e^x\left(\sum_{i=1}^{N} q_i x^i + 1\right). \qquad [4.20]$$

Substituting Eq. (4.20) into Eq. (4.8) for N = 3, we obtain the following matrices with the help of Matlab program

$$D = \begin{pmatrix} 26.1182 & 36.1878 \\ 36.1878 & 54.7144 \end{pmatrix}, \quad b = \begin{pmatrix} 25.0219 \\ 34.9641 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \end{pmatrix}. \qquad [4.21]$$

And the approximate solution is.

$$\tilde{y} = -0.06457x^3 - 0.86856x^2 + 0.9x + 1. \qquad [4.22]$$

The exact solution is given by.

$$\boldsymbol{y_{exact}} = -3.7246e^{-\frac{x}{2\sqrt{2}}}(1.\mathrm{BesselJ}[-\frac{1}{\sqrt{2}}, 2\sqrt{e^x}] + 0.2591\mathrm{BesselJ}[\frac{1}{\sqrt{2}}, 2\sqrt{e^x}]). \qquad [4.23]$$

The figure below depicts the graph of approximate with exact solutions and error between them for $N = 3$

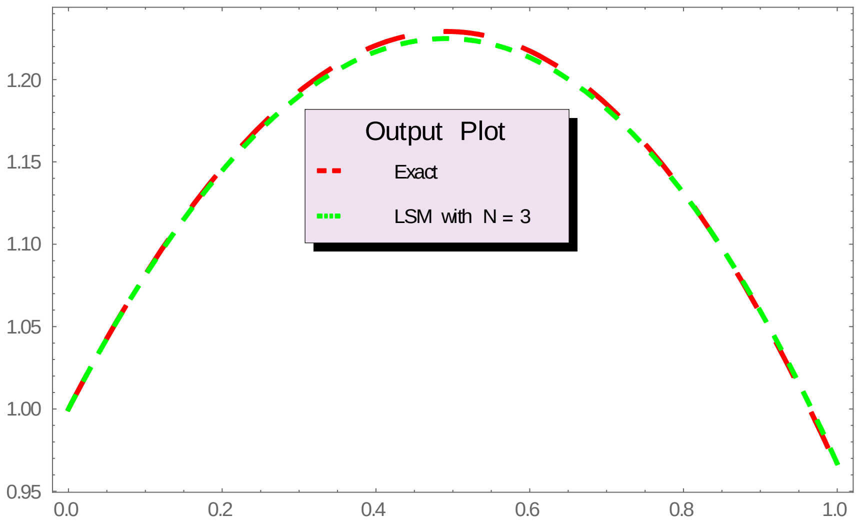

Figure 7: Example of second order with CLSM for *N*=3

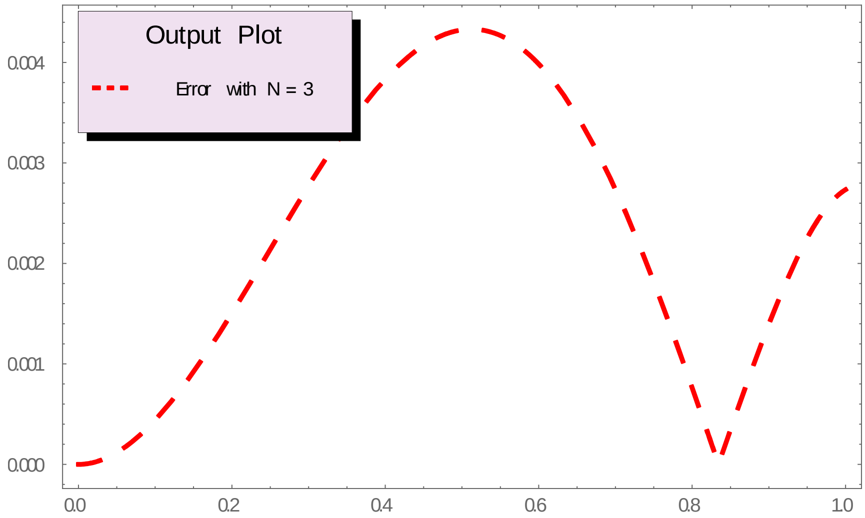

Figure 8: Error of second-order ODE for N = 3

For $N = 5$, we obtain the following matrices

$$D = \begin{pmatrix} 26.12 & 36.19 & 46.96 & 58.13 \\ 36.19 & 54.71 & 74.13 & 94.10 \\ 46.96 & 74.13 & 103.8 & 135.1 \\ 58.13 & 94.1 & 135.1 & 179.0 \end{pmatrix}, \quad b = \begin{pmatrix} 25.0 \\ 34.96 \\ 45.45 \\ 56.25 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}. \quad [4.28]$$

And the approximate solution is.

$$\tilde{\boldsymbol{y}} = 0.0646\boldsymbol{x}^{\mathbf{5}} - 0.0787\boldsymbol{x}^{\mathbf{4}} - 0.0999\, x^3 - 0.8218x^2 + 0.9x + 1. \quad [4.29]$$

We know that the exact solution is.

$$\boldsymbol{y_{exact}} = -3.7246e^{-\frac{x}{2\sqrt{2}}}(1.\mathrm{BesselJ}[-\frac{1}{\sqrt{2}}, 2\sqrt{e^x}] + 0.2591\mathrm{BesselJ}[\frac{1}{\sqrt{2}}, 2\sqrt{e^x}]).$$

The figure below depicts the graph of approximate with exact solutions and error between them for $N = 3$

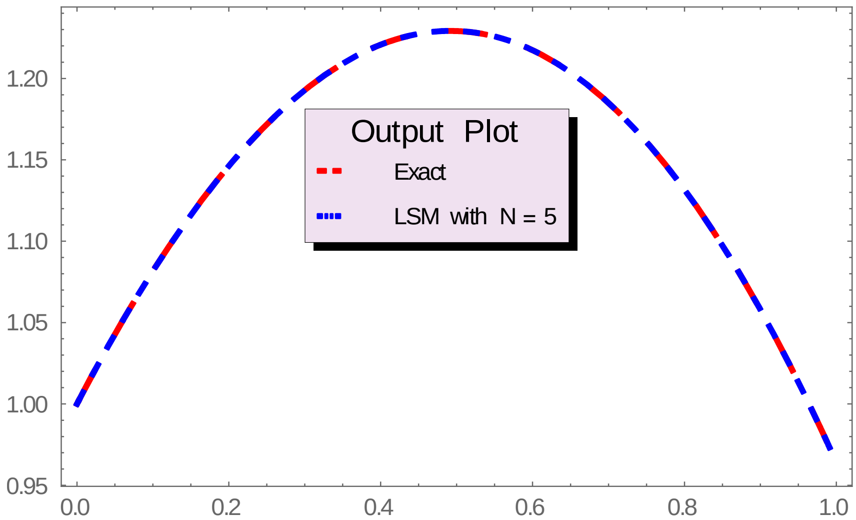

Figure 9: Example of second order with CLSM for *N*=5

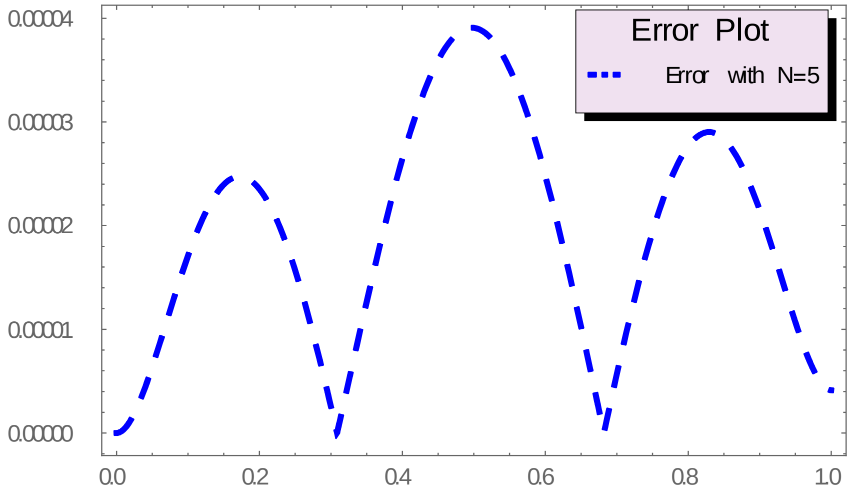

Figure 10: Error of second order for N=5

The figures below depict the comparison between the graph of exact with approximate solutions, and error for $N = 3$ and that of $N = 5$.

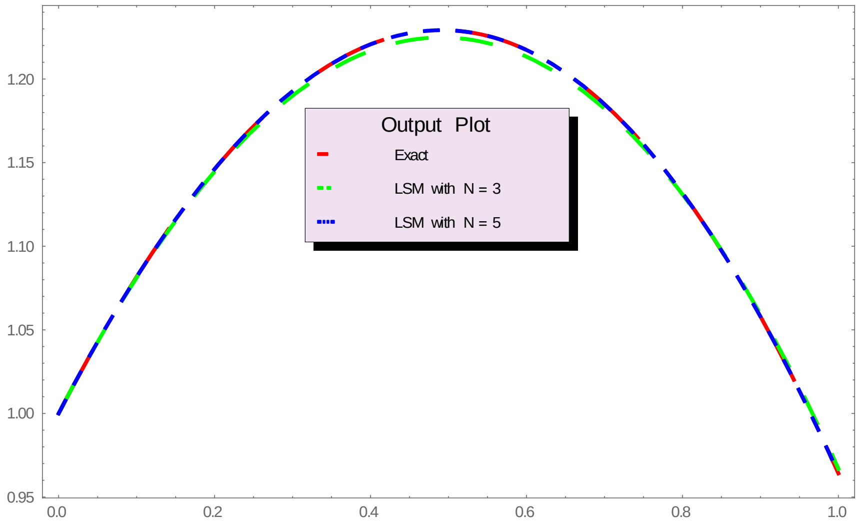

Figure 11: Example of second order with LSM for N=3 and N=5

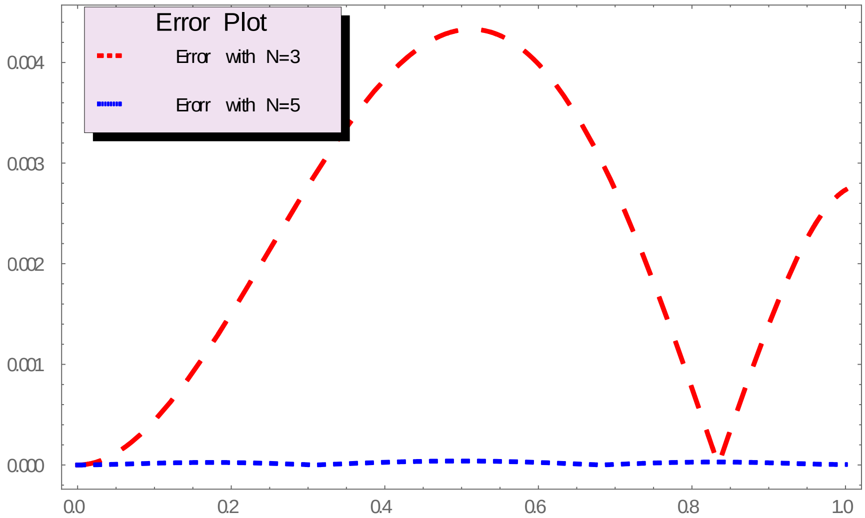

Figure 12: Error of second order for $N = 3$ and $N = 5$

Table 2: Data for the example of second-order ODE with errors for *N*=3 and *N*=5

x	y exact	y approx. for N = 3	y approx. for N = 5	Errors for N = 3	Errors for N = 5
0	1	1	1	0	0
0.1	1.0817	1.0813	1.0817	0.0004	0.000017
0.2	1.1463	1.1447	1.1462	0.0015	0.000023
0.3	1.1929	1.1901	1.11929	0.0028	2.5623×10^{-6}
0.4	1.2207	1.2169	1.2208	0.0038	0.000026
0.5	1.2291	1.2248	1.2292	0.0043	0.000039
0.6	1.2174	1.2134	1.2134	0.0040	0.000025
0.7	1.185	1.1823	1.1850	0.0027	5.6553×10^{-6}
0.8	1.1318	1.1311	1.1318	0.0008	0.000028
0.9	1.058	1.0594	1.0580	0.0014	0.000021
1.0	0.9613	0.9641	0.9669	0.0027	4.121×10^{-6}

5. Conclusion

This paper investigates numerical methods for solving differential equation. The continuous least square method (CLSM) was considered to find the best approximation by solving differential equations. The continuous least square method (CLSM) was developed together with the L_2 norm to find the best approximation with minimal error by solving differential equations. We apply the continuous (CLSM) method on both first-order differential equations with N = 3, 5 and second differential equations with N = 3, 5. The numerical results obtained proof to be the best approximation with minimum error. Explicit results are supported by comparative illustrations and examples treated with MATLAB and Wolfram Mathematica 11.

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