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On some Generalized Nörlund Ideal Convergent Sequence Spaces

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A B S T R A C T

In this paper, some new Ideal convergent sequence spaces $c^{I}_{(N,p,q)}$,$(c_0)^{I}_{(N,p,q)}$ and $(\ell_\infty)^{I}_{(N,p,q)}$ that are related to the $(N,p,q)-$ summability method, are introduced and some topological properties of these spaces and some inclusion relations and results are determined.

1. INTRODUCTION

We denote the space of all real valued sequences byω. Each vector subspace ofω is called as a sequence space as well. The spaces of all bounded, convergent and null sequences are denoted by ℓ_∞, c and c_0, respectively. By ℓ_1, ℓ_p, cs, cs_0 and bs, we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent, convergent to zero and bounded series, respectively; where$1 < p < \infty$.

A linear topological space λ is called a K-space if each of the map $\rho_i : \boldsymbol{\lambda} \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0,1,2,3,\dots\}$. A K-space λ is called an FK-space if λ is a complete linear metric space. If an FK-space has a normable topology then it is called a BK-space, (ABFB 2005). If λ is an FK-space, $\Phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have AK property, where (e^k) is a sequence whose only term in k^{th} place is 1 the others are zero for each $k \in \mathbb{N}$ and $\Phi=\text{span}\{e^k\}$. If Φ is dense in λ, then λ is called AD-space, thus AK implies AD.

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of real or complex numbers, where $n, k \in \mathbb{N}$. For every sequence $X = (X_k) \in \boldsymbol{\lambda}$ the sequence $Ax = Ax = ((Ax)_n) \in \boldsymbol{\mu}$ is called A-transform of x, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k . \quad (1)$$

Then, a defines a matrix mapping from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ and we show it by writing $A: \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu}$.

By $A \in (\lambda : \mu)$, we denote the class of all matrices A such that $A : \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu}$ if and only if the series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \boldsymbol{\lambda}$, and we have $Ax = ((Ax)_n)$ belongs to $\boldsymbol{\mu}$ for all $x \in \boldsymbol{\lambda}$. A sequence x is said to be A-summable to l and is called as the A-limit of x.

Let $\boldsymbol{\lambda}$ be a sequence space and A be an infinite matrix. The matrix domain λ_A of A in $\boldsymbol{\lambda}$ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : \ Ax \in \lambda\}$$

Which is a sequence space.

Let (t_k) be a nonnegative real sequence with $t_0 > 0$ and $T_n = \sum_{k=0}^{n} t_k$ for all $\in \mathbb{N}$. Then, the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a_{nk}^t)$ as follows

$$a_{nk}^t = \begin{cases} \frac{t_{n-k}}{T_n} & , \ 0 \leq k \leq n \\ 0 & , \ k > n \end{cases} \quad (2)$$

for every $k, n \in \mathbb{N}$. It is know that the Nörlund matrix N^t is a Teoplitz matrix if and only if $\frac{t_n}{T_n} \longrightarrow 0$, as $n \longrightarrow \infty$. Furthermore, if we take $t = e = (1,1,1,\dots)$, then the Nörlund matrix N^t is reduced to Cesàro mean C_1 of order one and if we choose $t_n = A_n^{r-1}$ for every $n \in \mathbb{N}$, then the N^t Nörlund mean becomes Cesàro mean C_r of order r, where $r > -1$ and

$$A_n^t = \begin{cases} \frac{(r+1)(r+2)\dots(r+n)}{n!} & , \ n = 1,2,3,\dots \\ 0 & , \ n = 0 \end{cases}$$

Let $t_0 = D_0 = 1$ and define D_n for $n \in \{1,2,3,\dots\}$by

$$D_n = \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 1 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & 1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix} (3)$$

With $D_1 = t_1$, $D_2 = (t_1)^2 - t_2$, $D_3 = (t_3)^3 - 2t_1t_2 + t_3 \dots \dots$ then the inverse matrix $U^t = (u_{nk}^t)$ of Nörlund matrix $N^t$was defined by Mears in (MFM 1943) for all $n \in \mathbb{N}$ as follows

$$u_{nk}^t = \begin{cases} (-1)^{n-k} D_{n-k} T_K & , \ 0 \leq k \leq n, \\ 0 & , \ k > n. \end{cases} \quad (4)$$

Definition 1.1. A family $I \subset 2^x$ of subset of a nonempty set X is said to be an ideal in X if

i) $\emptyset \in X$,

ii) For $A, B \in I$ imply $A \cup B \in I$,

iii) $A \in I, B \subset A$ imply $B \in I$.

The ideal I of X is said to be non-trivial if and only if $I \neq 2^X$. The non-trivial ideal $I \subset 2^x$is called an admissible ideal in X if and only if it contains $\{\{y\} : y \in X\}$. A non-trivial ideal I is called maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Definition 1.2. Let $I \subset 2^x$ be an ideal on X. The non-empty family of sets $F(I) \subset 2^x$ is called Filter on X corresponding to I if and only if

i) $\emptyset \notin F(I)$,

ii) For $A, B \in F(I)$ imply $A \cap B \in F(I)$,

iii) For each $A \in F(I)$ and $A \subset B$ implies $B \in F(I)$.

For each ideal I, there is a Filter F(I) corresponding to I. that is ,the following set F(I) is called filter according to the ideal I

$F(I) = \{K \subset 2^X : \ K^c \in I\}$,

where $K^c = X \backslash K = X - K$

Definition 1.3. The sequence $x = (x_n)_{n \in \mathbb{N}} \in \omega$ is called ideal convergent or I-convergent to a number L if for every $e > 0$

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I$$

And if is denoted by

$$I - \lim x_n = L .$$

The space of all I-convergent sequences to L is denoted by c^I as follow;

$$c^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I\},$$

for some $L \in \mathbb{C}$. (See KPW 2014, STT 2000, STB 2004, OT 2012, OTMD 2012).

Definition 1.4. The sequence $x = (x_n)_{n\in\mathbb{N}} \in \omega$ is said to be I-null if $L = 0$. In this case it is denoted by

$$I - \lim x_n = 0$$

The space of all I-null sequences is defined by c_0^I as

$$c_0^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : |x_k| \geq \varepsilon\} \in I\}$$

(See KPW 2014, STT 2000, STB 2004, OT 2012, OTMD 2012).

Definition 1.5. A sequence $x = (x_n)_{n\in\mathbb{N}} \in \omega$ is said to be I-bounded if there exist a real constant $M \geq 0$ such that

$$\{k \in \mathbb{N} : |x_k| \geq M\} \in I$$

(TBC 2005)

Definition 1.6. Let X be a linear space. A function $g : X \to \mathbb{R}$ is called a paranorm if for all $x, y, z \in X$;

i) $g(x) = 0$ if $x = \theta$,
ii) $g(-x) = g(x)$,
iii) $g(x + y) \leq g(x) + g(y)$,
iv) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda (n \to \infty)$ and $x_n, L \in X$ with $x_n \to L (n \to \infty)$ in the sense that $g(x_n - L) \to 0 (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda L) \to 0 (n \to \infty)$.

Definition 1.7. A sequence space X is called solid or normal if $x = (x_k) \in X$ implies $\alpha x = (\alpha_k x_k) \in X$ for all sequence of scalars $\alpha = (\alpha_k)$ with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$,(TBC 2005)

Definition 1.8. A sequence space X is called monotone if it contains the canonical pre-images of all its step-spaces, (TBC 2005)

Let $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ and E be a sequence space. A K- step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_n) \in E\}$. A canonical preimage of a sequence $x_{k_n} \in \lambda_K^E$ is a sequence $y = (y_n) \in \omega$ defined as

$$y_n \begin{cases} x_n & , \quad \text{if } n \in K \\ 0 & , \quad \text{otherwise} \end{cases}$$

A canonical perimage of step space λ_K^E is a set of canonical preimage of all the elements in λ_K^E if and only if is a canonical perimage of some $x \in \lambda_K^E$ see (HBT 2014).

Lemma 1.9. The sequence space X is solid implies that X is monotone, (see KPK 2009 p.53).

2.GENERALIZED WEIGHTED NORLUND IDEAL CONVERGENCE

Let $p = (p_k)$ and $q = (q_k)$ be two increasing sequences of non-zero real constant which satisfy

$$P_n = p_1 + p_2 + \cdots + p_n , P_{-1} = p_{-1} = 0,$$
$$Q_n = q_1 + q_2 + \cdots + q_n , Q_{-1} = q_{-1} = 0$$

Now, we define the Cauchy product of the sequences P_n and Q_n, as follow

$$R_n = (p_n) * (q_n) = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k$$

Then, the series $\sum_k x_k$ or any sequence $x = (x_k)$ is summable to any point L by generalized Nörlund method which is denoted by $x_k \to L(N,p,q)$ if

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} x_k = L.$$

This is obvious that when we take $p_n = 1$ for each $n \in \mathbb{N}$, then we Nörlund method. (See OTFB 2016). Since we take $p_n = q_n = 1$ for each $n \in \mathbb{N}$, then we approach Cesaro method.

The matrix $A = (\alpha_{nk})$ in (N,p,q)-summability is defined by

$$\alpha_{nk} = \begin{cases} \frac{p_k q_{n-k}}{R_n} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

In this paper, we construct the new I-convergent sequence spaces related to the (N,p,q)- summability method. Now, by $c^I_{(N,p,q)}$, $(c_0)^I_{(N,p,q)}$ and $(\ell_\infty)^I_{(N,p,q)}$, we define generalized weighted Nörlund I-convergent, generalized weighted Nörlund I-null and generalized weighted Nörlund I-bounded sequence spaces, respectively. First we give some topological properties of these spaces. Then, we derive some inclusion relations and results.

A sequence $x = (x_k)$ is said to be generalized weighted Nörlund ideal convergent if for every $\varepsilon > 0$

$$N(\epsilon) = \left\{ n \in N : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| \ge \varepsilon \right\} \in I$$

And the set of all generalized weighted Nörlund $I-$ convergent , generalized weighted Nörlund I $-$null and generalized weighted Nörlund I $-$bounded sequence spaces are defined as follows ;

$$c^I_{(N,p,q)} =$$

$$\left\{ x = (x_k) \in \omega : \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| \ge \varepsilon \right\} \in I \right\}$$

$$(c_0)^I_{(N,p,q)} =$$

$$\left\{ x = (x_k) \in \omega : \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| \ge \varepsilon \right\} \in I \right\}$$

$$(l_\infty)^I_{(N,p,q)} =$$

$$\left\{ x = (x_k) \in \omega : \left\{ n \in \mathbb{N} : \exists M > 0 \ni \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k| > M \right\} \in I \right\}$$

Theorem 2.1. The spaces $c^I_{(N,p,q)}$, $(c_0)^I_{(N,p,q)}$, $(l_\infty)^I_{(N,p,q)}$ are linear spaces

Proof. We shall prove the result for the space $c^I_{(N,p,q)}$. Let $x = (x_k), y = (y_k) \in c^I_{(N,p,q)}$ and $\alpha, \beta \in \mathbb{C}$ are given. Then we have the following for given every $\epsilon > 0$

We denote

$$A(\epsilon) = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L_1| \ge \frac{\varepsilon}{2} \right\} \in I$$

$$B(\epsilon) = \left\{ n \in \mathbb{N} : \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |y_k - L_2| \ge \frac{\varepsilon}{2} \right\} \in I$$

for some $L_1, L_2 \in \mathbb{C}$.

Now, we write the following inequality

$$\frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|$$

$$\le \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} (|\alpha||x_k - L_1| + |\beta||y_k - L_2|)$$

$$\le |\alpha| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L_1| + |\beta| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |y_k - L_2|$$

Then,by using the above inequality we derive

$$\left\{n\in N:\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}(\alpha x_k+\beta y_k)-(\alpha L_1+\beta L_2)|\geq\epsilon\right\}$$

$$\subseteq\left\{n\in\mathbb{N}:|\alpha|\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k-L_1|\geq\frac{\epsilon}{2}\right\}$$

$$\cup\left\{n\in\mathbb{N}:|\beta|\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|y_k-L_2|\geq\frac{\epsilon}{2}\right\}$$

$$\subseteq A(\epsilon)\cup B(\epsilon)\in I$$

Then this completes the proof. The proof for the spaces $c^I_{(N,p,q)}$ and $(l_\infty)^I_{(N,p,q)}$ follow similarly.

Theorem 2.2. The spaces $c^I_{(N,p,q)}$, $(c_0)^I_{(N,p,q)}$, $(l_\infty)^I_{(N,p,q)}$ are para-normed spaces with the para-norm

$$g(x)=\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|$$

Proof. Since we have similar proof for $c^I_{(N,p,q)}$, $(c_0)^I_{(N,p,q)}$, $(l_\infty)^I_{(N,p,q)}$,we give only the proof for $c^I_{(N,p,q)}$. It is trivial that if $x=(x_k)=0$ then $g(x)=0$.for $x=(x_k)\neq 0$ then $g(x)\neq 0$, we have that

i) For all $x\in c^I_{(N,p,q)}$

$$g(x)=\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|\geq 0$$

ii) For all $x\in c^I_{(N,p,q)}$

$$g(-x)=\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|-x_k|$$

$$=\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|=g(x)$$

iii) For every $x,y\in c^I_{(N,p,q)}$

$$g(x+y)=\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k-y_k|$$

$$\leq\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|$$

$$+\sup_{n\epsilon\mathbb{N}}\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|y_k|$$

$$=g(x)+g(y).$$

iv) Let (λ_n) is a sequence of scalars with $\lambda_n\to\lambda(n\to\infty)$ and $x_n\in c^I_{(N,p,q)}$

such that

$$\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|\to L(n\to\infty),$$

in the sense that

$$g\left(\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|\to L\right)\to 0(n\to\infty)\quad.$$

Therefore,

$$g\left(\lambda_n\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|-\lambda L\right)\leq$$

$$g\left(\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|(\lambda_n-\lambda)\right)$$

$$+g\left(\lambda\left(\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|-L\right)\right)$$

Then it is obvious that

$$\lambda_n\frac{1}{R_n}\sum_{k=0}^{n}p_k\,q_{n-k}\,|x_k|\to\lambda L(n\to\infty).$$

This is completes the proof.

Theorem 2.3. The space $c^I_{(N,p,q)}$ is solid and monotone.

Proof. Suppose that $x=(x_k)\in c^I_{(N,p,q)}$ and (α_k) be a sequence of scalars with $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$. Then notice that

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|\alpha_k x_k| \le \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|\alpha_k||x_k|$$

$$\le \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|x_k|.$$

Furthermore,

(12) $$\left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|\alpha_k x_k| \ge \varepsilon\right\}$$

$$\subseteq \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|x_k| \ge \varepsilon\right\}$$

Then by using (12) we derive $(\alpha_k x_k) \in c^I_{(N,p,q)}$. This completes the proof.

Theorem 2.4. $c^I_{(N,p,q)}$ is a closed subset of $(l_\infty)^I_{(N,p,q)}$.

Proof. Let's take a Cauchy sequence $x_k^{(n)}$ in $\in c^I_{(N,p,q)}$ such that $x^{(n)} \to x\ as\ n \to \infty$. We need to show that $x \in c^I_{(N,p,q)}$. Since $x_k^{(n)} \in c^I_{(N,p,q)}$ then there exist a sequence of complex number α_n such that

$$A = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|x_k{}^{(n)} - \alpha_n| \ge \varepsilon\right\} \in I$$

(13)

Now, to give the proof , we need to mention that $\alpha_n \to x\ as\ n \to \infty$ and $(A')^c \in I$ whenever

$$A' = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\,|x_k - a| \ge \varepsilon\right\}$$

Since $x^{(n)}$ is a Cauchy sequence in $c^I_{(N,p,q)}$. We can write for a given $\epsilon > 0$, there exist $k_0 \in \mathbb{N}$ such that

$$\frac{1}{R_n}\sum_{k=0}^{n} p_k\, q_{n-k}\left|x_k^{(n)} - x_k^{(m)}\right| < \frac{\epsilon}{3} \quad for\ all\ m,n \ge k_0$$

Let us define the followings sets for $\varepsilon > 0$ as:

$$A_1 = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - x_k^{(m)}\right| < \frac{\epsilon}{3}\right\}$$

$$A_2 = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - a_n\right| < \frac{\epsilon}{3}\right\}$$

$$A_3 = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - a_m\right| < \frac{\epsilon}{3}\right\}$$

For all $m,n \ge k_0$ whenever $A_1^c,\ A_2^c,\ A_3^c \in I$. Then we have

$$\left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}|a_n - a_m| < \epsilon\right\} \supseteq$$

$$\left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - x_k^{(m)}\right| < \frac{\epsilon}{3}\right\}$$

$$\cap \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - a_n\right| < \frac{\epsilon}{3}\right\}$$

$$\cap \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - a_m\right| < \frac{\epsilon}{3}\right\}$$

We can see that (a_n) is a Cauchy sequence in $\mathbb{C}$ and convergent to the scalar a as $n \to \infty$.

Now, for the last needed let's take $0 < \delta < 1$. Then we need to show that if

$$A' = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}|x_k - a| < \delta\right\}$$

Then $(A')^c \in I$. Since

$\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - x_k\right| \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that

$$E_1 = \left\{n\in\mathbb{N}: \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left|x_k^{(n)} - x_k\right| < \frac{\delta}{3}\right\}$$

Which implies that $(E_1)^c \in I$ for all $n \geq n_0$. And we already have from the first part that

$$E_2 = \left\{ n \in \mathbb{N}: \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |a_n - a| < \frac{\delta}{3} \right\}$$

Which gives us $(E_2)^c \in I$ for all $n \geq n_0$. Since the set $A \in I$ defined as in (13) δ instead of ε, then we have a subset $E_3 \subset \mathbb{N}$ such that $(E_3)^c \in I$ whenever,

$$E_3 = \left\{ n \in \mathbb{N}: \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \left| x_k^{(n)} - a_n \right| < \frac{\delta}{3} \right\}$$

Then we may easily say that $(A')^c \supseteq E_1 \cap E_2 \cup E_3$. Then by the definition of filter on the ideal that we can say $C^I_{(N,p,q)} \subset (l_\infty)^I_{(N,p,q)}$. This completes the proof.

Theorem 2.5. The inclusions

$(c_0)^I \subset c^I_{(N,p,q)} \subset (l_\infty)^I$ are proper.

Proof. Let's take a sequence

$x = (x_k) \in (c_0)^I$. Then we have

$$\{n \in \mathbb{N}: |x_n| \geq \epsilon\} \in \mathrm{I}$$

Since $c_0 \subset c_{(N,p,q)} \subset l_\infty$ which give us that $x = (x_k) \in c^I_{(N,p,q)}$ implies

$$\left\{ n \in \mathbb{N}: \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| \geq \epsilon \right\} \in I$$

Now, let us define the following sets

$$A_1 = \{n \in \mathbb{N}: |x_n - L| < \epsilon\}$$

$$A_2 = \left\{ n \in \mathbb{N}: \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} |x_k - L| < \epsilon \right\}$$

Such that $A_1^c, A_1^c \in I$. Since

$$\ell_\infty = \{x = (x_n) \in \omega: sup_n |x_n| < \infty\}$$

When we take supremum over n then we get $A_1^c \subset A_1^c$. Then we conclude as $(C_0)^I \subset C^I_{(N,p,q)} \subset (l_\infty)^I$.

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