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COMMUTATIVITY ASSOCIATED WITH EULER SECOND-ORDER DIFFERENTIAL EQUATION

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Abstract

We study commutativity and the sensitivity of the second-order Euler differential equation. The necessary and sufficient conditions for commutativity of the second-order Euler differential equation are considered. Moreover, the stability, the robustness, and the effect due to disturbance on the second-order Euler linear time-varying system (LTVS) are investigated. An example is given to support the results. The results are well verified using the Matlab Simulink toolbox.

1. Introduction

The ordinary differential equations (ODEs) study the behavior and changes that occur in physical systems. ODEs play an important role in

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science and engineering. But there is a problem in finding solutions of some difficult differential equations. Authors like Ibrahim [1]; Ibrahim [19]; Ibrahim and Isah [2]; Isah and Ibrahim [6] introduced numerical methods for solving ODEs, partial differential equations (PDEs), and fractional differential equations (FDEs).

When the sensitivity, stability, linearity, noise disturbance and robustness effects are considered, the change of the order of connections without changing the main function of the total systems, commutativity may lead to positive results. Therefore, commutativity is very important from the practical point of view.

The first commutativity appeared in 1977 for the first-order time-varying systems (Marshall [11]), and then the results therein are extended to higher-order systems (Koksal [7-9]; Saleh [15]; Koksal and Koksal [10]; Ibrahim and Koksal [4]). The nonzero initial conditions (ICs) and their effects on the sensitivity have been studied in Salisu [16], while the realization of a fourth-order LTVSs with nonzero ICs by cascaded two second-order commutative pairs was introduced in Ibrahim and Koksal [5]. Ibrahim and Koksal [3] presented the decomposition of fourth-order LTVSs, while the extension and its application to Euler LTVSs were studied in Salisu and Abedallah [17]; Ibrahim [18].

Rababah and Ibrahim [12-14] came up with a numerical approximative process for degree reduction of curves and surfaces which could be used to solve complex ODEs, PDEs, and FDEs. In this study, the commutativity and the sensitivity of the second-order Euler LTVS are considered. The results are illustrated and supported by an example.

2. System Description

Consider the cascade connection of second-order systems A and B described as

$$A: a_2(t)\, y''_A(t) + a_1(t)\, y'_A(t) + a_0(t)\, y_A(t) = x_A(t), \tag{1}$$

$$B: b_2(t)\, y''_B(t) + b_1(t)\, y'_B(t) + b_0(t)\, y_B(t) = x_B(t), \tag{2}$$

where $a_2(t) \neq 0$ and $b_2(t) \neq 0$. Also, $a_i, b_i, x_A, x_B \in P[t_0, \infty)$. The connections are abbreviated as *AB* or *BA* according to their sequence of connections.

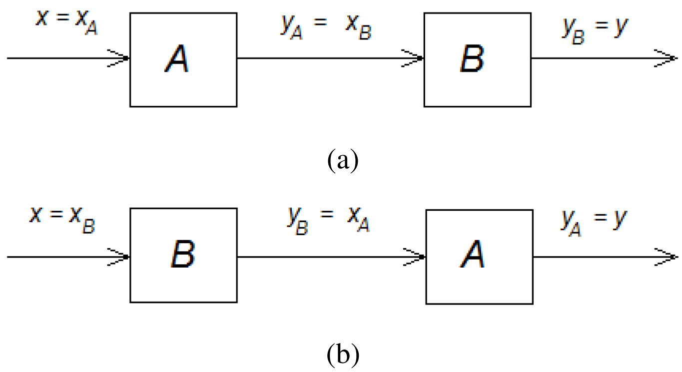

Figure 1. Cascade connection of the differential systems *A* and *B*.

The propose is to find the subsystems *A* and *B* such that the connections *AB* and *BA* are equivalent (the case in which *A* and *B* are called *commutative subsystems*); the results found are expressed in theorems stated in the next section.

3. Main Results

Theorem 1 (See Koksal [7]). *The formula for a second-order LTVS A to be commutative with another LTVS B under zero ICs is that the coefficients of B be*

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & f_{32} & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \; f_{32} = \frac{1}{4}[a_2^{-0.5}(2a_1 - a_2')]; \tag{3a}$$

$$-a_2^{0.5} \frac{d}{dt}[a_0 - f_{32}^2 - a_2^{0.5} f_{32}']k_1 = 0, \tag{3b}$$

where k_2, k_1, k_0 *are constants and satisfy* (3b).

Theorem 2. *The following are conditions of the commutativity for second-order LTVS A with nonzero ICs with another second or lower-order LTVS B*:

(i) *Equations* (3a) *and* (3b) *must be satisfied*.

(ii) *The ICs at the initial time* (*IT*) $t_0 \leq t$ *must hold*:

$$\left\{\binom{2}{m}\begin{bmatrix} 1 & 0 \\ -A_2^{-1}A_1 & A_2^{-1} \end{bmatrix} - \binom{m}{2}\begin{bmatrix} 0 & 1 \\ B_2^{-1} & -B_2^{-1}B_1 \end{bmatrix}\right\}\begin{bmatrix} Y_A \\ Y_B \end{bmatrix} = [0], \tag{4}$$

where

$$Y_A = [y_A(t),\ y'_A(t)]^T,$$

$$Y_B = [y_B(t),\ y'_B(t)]^T$$

and the matrices $A_1\,(A_2,\, B_1,\, B_2)$ *are described by their entries* $a'_{ij}\,(a''_{ij},\, b'_{ij},\, b''_{ij})$, *respectively*:

$$a'_{ij} = \sum_{s=\max(0,\, i-j)}^{i-1} \frac{(i-1)!}{s!(i-1-s)!} a^s_{j-i+s}; \quad i = 1,\, m, \quad j = 1,\, 2,$$

$$a''_{ij} = \sum_{s=0}^{i-j} \frac{(i-1)!}{s!(i-1-s)!} a^s_{j-i+n+s}; \quad i = 1,\, m, \quad j = 1,\, m;$$

$$= 0 \ \textit{for} \ i = 1,\, \ldots,\, m-1, \quad j = i+1,\, \ldots,\, m,$$

$$b'_{ij} = \sum_{s=\max(0,\, i-j)}^{i-1} \frac{(i-1)!}{s!(i-1-s)!} b^s_{j-i+s}; \quad i = 1,\, 2, \quad j = 1,\, m,$$

$$b''_{ij} = \sum_{s=\max(0,\, i-j-m)}^{i-j} \frac{(i-1)!}{s!(i-1-s)!} b^s_{j-i+m+s}; \quad i = 1,\, 2, \quad j = 1,\, \ldots,\, i;$$

$$= 0 \ \textit{for} \ i = 1, \quad j = i+1,\, \ldots,\, 2. \tag{5}$$

4. Example

In this section, we make use of the formula and conditions obtained from the previous section and illustrate the commutativity of second-order LTVS.

Example 1. Consider the following second-order Euler LTVS:

$$A: t^2 y_A''(t) + \sqrt{2}t\, y_A'(t) + \frac{17}{11} y_A(t) = x_A(t). \tag{6}$$

By applying the coefficients of equation (6) to equation (3a), we obtain

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \sqrt{2}t & t & 0 \\ \frac{17}{11} & \frac{1}{4}\left(\frac{-2t + 2\sqrt{2}t}{t}\right) & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \tag{7}$$

where k_2, k_1 and k_0 are constants.

Considering the matrix in equation (7) at $k_1 = 0$, we have

$$B: t^2 k_2 y_B''(t) + \sqrt{2}t\, k_2 y_B'(t) + \left(k_0 + \frac{17}{11} k_2\right) y_B(t) = x_B(t). \tag{8}$$

Substituting the coefficients of equation (6) in equation (3b), we have

$$k_0 \to 1 - k_2. \tag{9}$$

For commutativity of A and B with nonzero ICs to exist, the equation below must be satisfied

$$y_A' = -\frac{3\sqrt{2}}{11} y_A. \tag{10}$$

Consider a sinusoid of amplitude 200, frequency 25, and phase $\frac{\pi}{30}$ rad with ODE 23 [Bogacki-Shampine] as the solver. For $k_2 = k_0 = 0.5$ and $k_1 = 0$, the IT at $t_0 = 1$ and the ICs as $y_A(1) = y_B(1) = 1$, $y_A'(1) = y_B'(1) = -\frac{3\sqrt{2}}{11}$, AB and BA (solid blue curves) give the same output. With response to sensitivity toward ICs, by changing $y_A(1) = -1$, $AB1$ (doted-dash red curve) and $BA1$ (dashed-green curve) deviated from each other, this is because equation (10) is not obeyed.

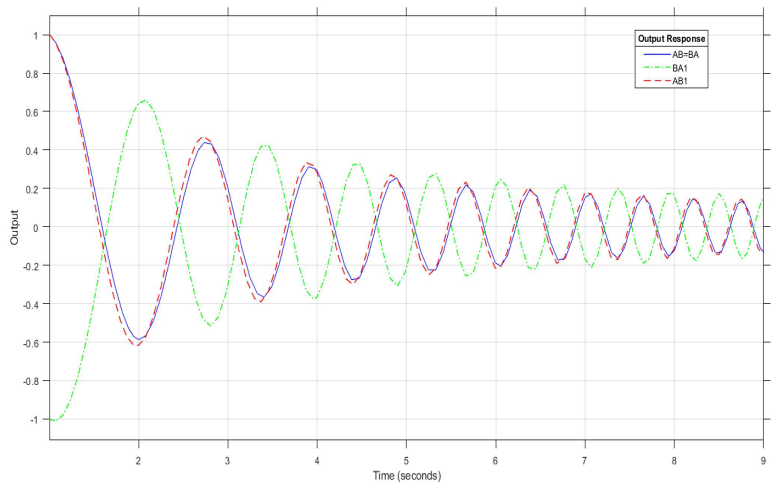

Figure 2. Simulation results for $k_2 = k_0 = 0.5$ and $k_1 = 0$.

5. Conclusion

This study shows the results for second-order Euler LTVS A cascaded connected with its commutative pairs of second-order LTVS B. The results obtained show that the subsystems A and B are commutative under some conditions and are very sensitive toward change in ICs. The results are verified by an example that is simulated by the Simulink toolbox of MATLAB. Furthermore, the findings can be extended to nonlinear LTVSs, PDEs, FDEs, systems of LTVSs and discrete LTVSs.

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